

Introduction to College Mathematics

Introduction to College Mathematics

*TANVIR PRICE, HOSTOS COMMUNITY
COLLEGE*



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PART II

MODULE I: BASIC OF SET

I. Set Theory

It is natural for us to classify items into groups, or sets, and consider how those sets overlap with each other. We can use these sets to understand relationships between groups, and to analyze survey data.

Basics

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

Set

A **set** is a collection of distinct objects, called **elements** of the set

A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

Example 1

Some examples of sets defined by describing the contents:

1. The set of all even numbers
2. The set of all books written about travel to Chile

Answers

Some examples of sets defined by listing the elements of the set:

1. $\{1, 3, 9, 12\}$
2. $\{\text{red, orange, yellow, green, blue, indigo, purple}\}$

A set simply specifies the contents; order is not important. The set represented by $\{1, 2, 3\}$ is equivalent to the set $\{3, 1, 2\}$.

Notation

Commonly, we will use a variable to represent a set, to make it easier to refer to that set later.

The symbol \in means “is an element of”.

A set that contains no elements, $\{\}$, is called the **empty set** and is notated \emptyset

Example 2

Let $A = \{1, 2, 3, 4\}$

To notate that 2 is element of the set, we'd write $2 \in A$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

Subset

A **subset** of a set A is another set that contains only elements from the set A , but may not contain all the elements of A .

If B is a subset of A , we write $B \subseteq A$

A **proper subset** is a subset that is not identical to the original set—it contains fewer elements.

If B is a proper subset of A , we write $B \subset A$

Example 3

Consider these three sets:

A = the set of all even numbers

$B = \{2, 4, 6\}$

$C = \{2, 3, 4, 6\}$

Here $B \subset A$ since every element of B is also an even number, so is an element of A .

More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$.

It is also true that $B \subset C$.

C is not a subset of A , since C contains an element, 3, that is not contained in A

Example 4

Suppose a set contains the plays “Much Ado About Nothing,” “MacBeth,” and “A Midsummer’s Night Dream.” What is a larger set this might be a subset of?

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

Try It Now

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

Union, Intersection, and Complement

Commonly sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

Union, Intersection, and Complement

The **union** of two sets contains all the elements contained in either set (or both sets). The union is notated $A \cup B$. More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both)

The **intersection** of two sets contains only the elements that are in both sets. The intersection is notated $A \cap B$. More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$.

The **complement** of a set A contains everything that is not in the set A . The complement is notated A' , or A^c , or *sometimes* $\sim A$.

Example 5

Consider the sets:

$A = \{\text{red, green, blue}\}$

$B = \{\text{red, yellow, orange}\}$

$C = \{\text{red, orange, yellow, green, blue, purple}\}$

Find the following:

1. Find $A \cup B$
2. Find $A \cap B$

3. Find $A^c \cap C$

Answers

1. The union contains all the elements in either set: $A \cup B = \{\text{red, green, blue, yellow, orange}\}$ Notice we only list red once.
2. The intersection contains all the elements in both sets: $A \cap B = \{\text{red}\}$
3. Here we're looking for all the elements that are *not* in set A and are also in C. $A^c \cap C = \{\text{orange, yellow, purple}\}$

Try It Now

Using the sets from the previous example, find $A \cup C$ and $B^c \cap A$

Notice that in the example above, it would be hard to just ask for A^c , since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections, or when we have a universal set in place.

Universal Set

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

A complement is relative to the universal set, so A^c contains all the elements in the universal set that are not in A .

Example 6

1. If we were discussing searching for books, the universal set might be all the books in the library.
2. If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
3. If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

Example 7

Suppose the universal set is U = all whole numbers from 1 to 9. If $A = \{1, 2, 4\}$, then $A^c = \{3, 5, 6, 7, 8, 9\}$.

As we saw earlier with the expression $A^c \cap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic – to force an order of operations.

Example 8

Suppose $H = \{\text{cat, dog, rabbit, mouse}\}$, $F = \{\text{dog, cow, duck, pig, rabbit}\}$, and $W = \{\text{duck, rabbit, deer, frog, mouse}\}$

1. Find $(H \cap F) \cup W$
2. Find $H \cap (F \cup W)$
3. Find $(H \cap F)^c \cap W$

Solutions

1. We start with the intersection: $H \cap F = \{\text{dog, rabbit}\}$. Now we union that result with W : $(H \cap F) \cup W = \{\text{dog, duck, rabbit, deer, frog, mouse}\}$
2. We start with the union: $F \cup W = \{\text{dog, cow, rabbit, duck, pig, deer, frog, mouse}\}$. Now we intersect that result with H : $H \cap (F \cup W) = \{\text{dog, rabbit, mouse}\}$

3. We start with the intersection: $H \cap F = \{\text{dog, rabbit}\}$. Now we want to find the elements of W that are not in $H \cap F$. $(H \cap F)^c \cap W = \{\text{duck, deer, frog, mouse}\}$

Venn Diagrams

To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the eighteenth century. These illustrations now called **Venn Diagrams**.

Venn Diagram

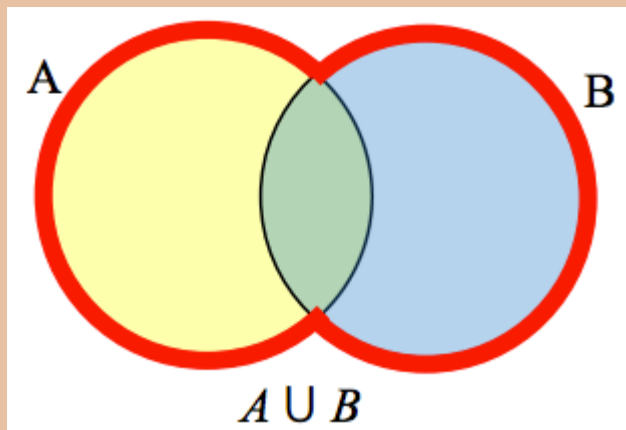
A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

Basic Venn diagrams can illustrate the interaction of two or three sets.

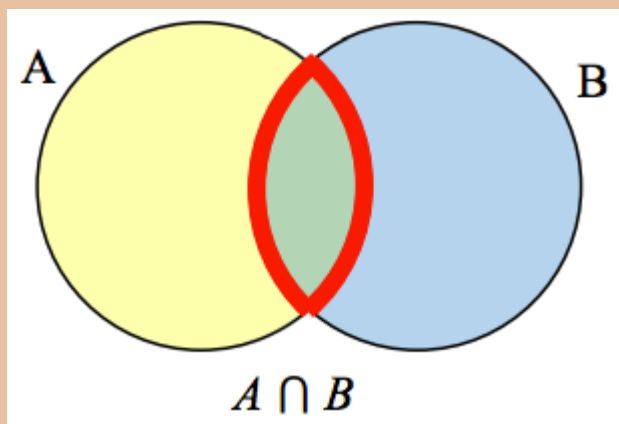
Example 9

Create Venn diagrams to illustrate $A \cup B$, $A \cap B$, and $A^c \cap B$

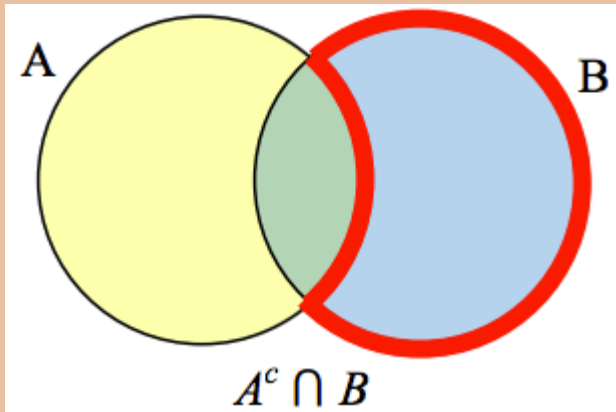
$A \cup B$ contains all elements in *either* set.



$A \cap B$ contains only those elements in both sets—in the overlap of the circles.



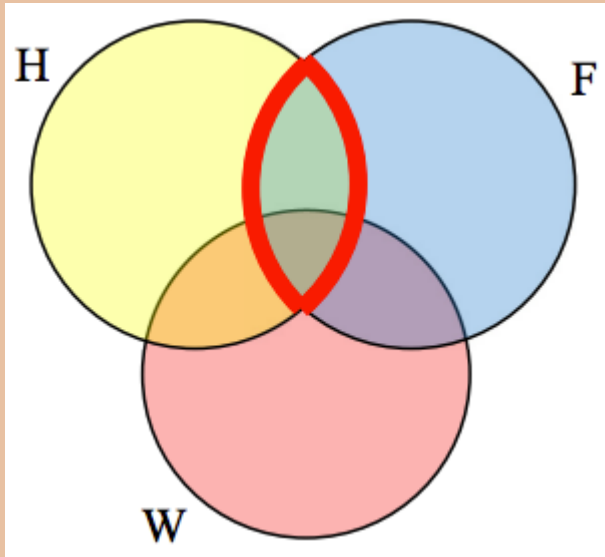
A^c will contain all elements not in the set A. $A^c \cap B$ will contain the elements in set B that are not in set A.



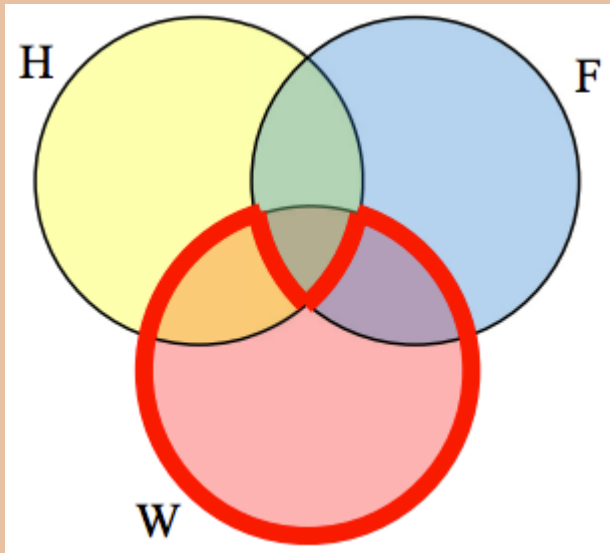
Example 10

Use a Venn diagram to illustrate $(H \cap F)^c \cap W$

We'll start by identifying everything in the set $H \cap F$

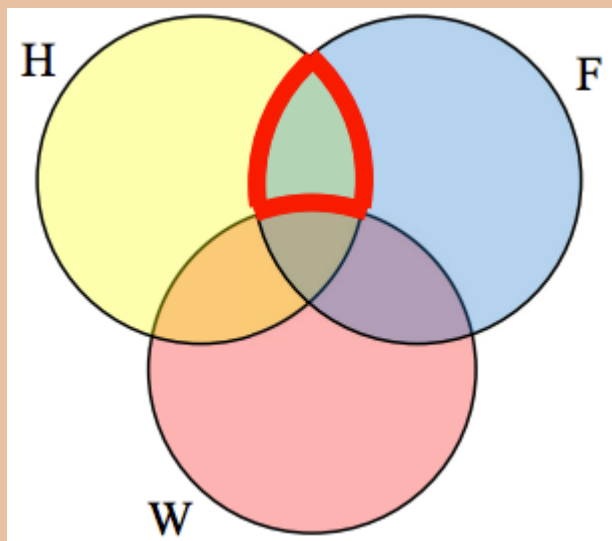


Now, $(H \cap F)^c \cap W$ will contain everything not in the set identified above that is also in set W.



Example 11

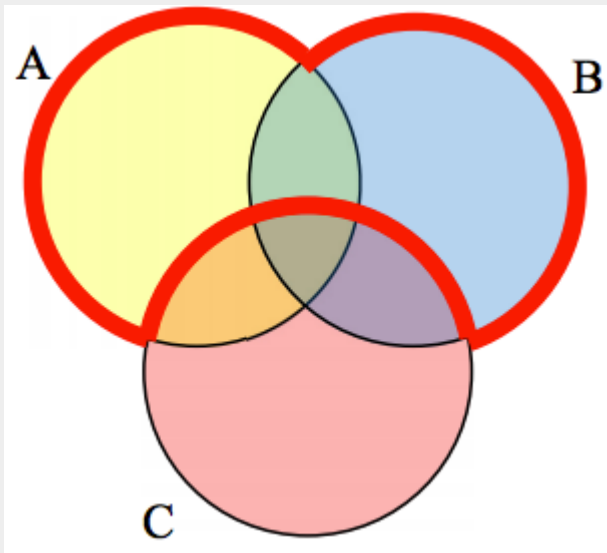
Create an expression to represent the outlined part of the Venn diagram shown.



The elements in the outlined set are in sets H and F, but are not in set W. So we could represent this set as $H \cap F \cap W^c$

Try It Now

Create an expression to represent the outlined portion of the Venn diagram shown



Cardinality

Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

Cardinality

The number of elements in a set is the cardinality of that set.

The cardinality of the set A is often notated as $|A|$ or $n(A)$

Example 12

Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$.

What is the cardinality of B ? $A \cup B$, $A \cap B$?

Answers

The cardinality of B is 4, since there are 4 elements in the set.

The cardinality of $A \cup B$ is 7, since $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$, which contains 7 elements.

The cardinality of $A \cap B$ is 3, since $A \cap B = \{2, 4, 6\}$, which contains 3 elements.

Example 13

What is the cardinality of P = the set of English names for the months of the year?

Answers

The cardinality of this set is 12, since there are 12 months in the year.

Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

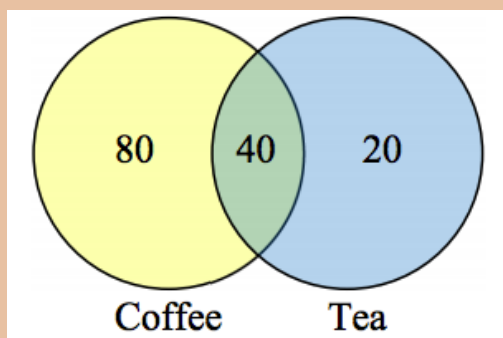
Example 14

A survey asks 200 people “What beverage do you drink in the morning”, and offers choices:

- Tea only
- Coffee only
- Both coffee and tea

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?

Answers



This

question can most easily be answered by creating a Venn diagram. We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.

We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.

$$200 - 20 - 80 - 40 = 60 \text{ people who drink neither.}$$

Example 15

A survey asks: "Which online services have you used in the last month?"

- Twitter
- Facebook
- Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?

Answers

Let T be the set of all people who have used Twitter, and F be the set of all people who have used Facebook. Notice that while the cardinality of F is 70% and the cardinality of T is 40%, the cardinality of $F \cup T$ is not simply $70\% + 40\%$, since that would count those who use both services twice. To find the cardinality of $F \cup T$, we can add the cardinality of F and the cardinality of T , then subtract those in intersection that we've counted twice. In symbols,

$$\begin{aligned}n(F \cup T) &= n(F) + n(T) - n(F \cap T) \\n(F \cup T) &= 70\% + 40\% - 20\% = 90\%\end{aligned}$$

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \cup T)^c$. Since the universal set contains 100% of people and the cardinality of $F \cup T = 90\%$, the cardinality of $(F \cup T)^c$ must be the other 10%.

The previous example illustrated two important properties

Cardinality properties

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A^c) = n(U) - n(A)$$

Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

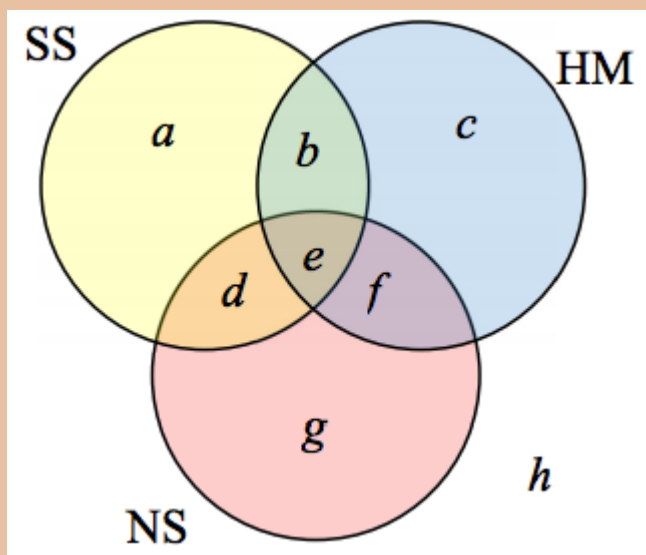
Example 16

Fifty students were surveyed, and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next quarter.

21 were taking a SS course	26 were taking a HM course
19 were taking a NS course	9 were taking SS and HM
7 were taking SS and NS	10 were taking HM and NS
3 were taking all three	7 were taking none

How many students are only taking a SS course?

Answers



It might help to look at a Venn diagram. From the given data, we know that there are 3 students in region e and 7 students in region h .

Since 7 students were taking a SS and NS course, we know that $n(d) + n(e) = 7$. Since we know there are 3 students in region 3, there must be $7 - 3 = 4$ students in region d .

Similarly, since there are 10 students taking HM and NS, which includes regions e and f , there must be $10 - 3 = 7$ students in region f .

Since 9 students were taking SS and HM, there must be $9 - 3 = 6$ students in region b .

Now, we know that 21 students were taking a SS course. This includes students from regions a , b , d , and e . Since we know the number of students in all but region a , we can determine that $21 - 6 - 4 - 3 = 8$ students are in region a .

8 students are taking only a SS course.

Try It Now

One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

43 believed in UFOs	44 believed in ghosts
25 believed in Bigfoot	10 believed in UFOs and ghosts
8 believed in ghosts and Bigfoot	5 believed in UFOs and Bigfoot
2 believed in all three	

How many people surveyed believed in at least one of these things?

PART III

MODULE 2: LOGIC

2. Logic

Logic is a systematic way of thinking that allows us to deduce new information from old information and to parse the meanings of sentences. You use logic informally in everyday life and certainly also in doing mathematics. For example, suppose you are working with a certain circle, call it “Circle X,” and you have available the following two pieces of information.

1. Circle X has radius equal to 3.
2. If any circle has radius r , then its area is πr^2 square units.

You have no trouble putting these two facts together to get:

3. Circle X has area 9π square units.

In doing this you are using logic to combine existing information to produce new information. Since a major objective in mathematics is to deduce new information, logic must play a fundamental role. This chapter is intended to give you a sufficient mastery of logic.

It is important to realize that logic is a process of deducing information correctly, *not* just deducing correct information. For example, suppose we were mistaken and Circle X actually had a radius of 4, not 3. Let's look at our exact same argument again.

1. Circle X has radius equal to 3.
2. If any circle has radius r , then its area is πr^2 square units.
3. Circle X has area 9π square units.

The sentence “Circle X has radius equal to 3.” is now untrue, and so is our conclusion “Circle X has area 9π square units.” But the logic is perfectly correct; the information was combined correctly, even if some of it was false. This distinction between correct logic and correct information is significant because it is often important

to follow the consequences of an incorrect assumption. Ideally, we want both our logic *and* our information to be correct, but the point is that they are different things.

In proving theorems, we apply logic to information that is considered obviously true (such as “Any two points determine exactly one line.”) or is already known to be true (e.g., the Pythagorean theorem). If our logic is correct, then anything we deduce from such information will also be true (or at least as true as the “obviously true” information we began with).

Statements

The study of logic begins with statements. A **statement** is a sentence or a mathematical expression that is either definitely true or definitely false. You can think of statements as pieces of information that are either correct or incorrect. Thus statements are pieces of information that we might apply logic to in order to produce other pieces of information (which are also statements).

Example 1

Here are some examples of statements. They are all true.

If a circle has radius r , then its area is πr^2 square units.

Every even number is divisible by 2.

$2 \in \mathbb{Z}$ (2 is an element of the set of integers (or more simply, 2 is an integer).)

$\sqrt{2} \notin \mathbb{Z}$ (The square root of 2 is not an integer.)

$\mathbb{N} \subseteq \mathbb{Z}$ (The set of natural numbers is a subset of the set of integers.)

The set $\{0,1,2\}$ has three elements.

Some right triangles are isosceles.

Example 2

Here are some additional statements. They are all false.

All right triangles are isosceles.

$$5 = 2$$

$\sqrt{2} \notin \mathbb{R}$ (The square root of 2 is not a real number.)

$\mathbb{Z} \subseteq \mathbb{N}$ (The set of integers is a subset of the set of natural numbers.)

$0, 1, 2 \cap \mathbb{N} = \emptyset$ (The intersection of the set $\{0,1,2\}$ and the natural numbers is the empty set.)

Example 3

Here we pair sentences or expressions that are not statements with similar expressions that are statements.

NOT Statements	Statements
Add 5 to both sides.	Adding 5 to both sides of $x - 5 = 37$ gives $x = 42$.
\mathbb{Z} (The set of integers)	$42 \in \mathbb{Z}$ (42 is an element of the set of integers.)
42	42 is not a number.
What is the solution of $2x = 84$?	The solution of $2x = 84$ is 42.

Example 4

We will often use the letters P, Q, R, and S to stand for specific statements. When more letters are needed we can use subscripts. Here are more statements, designated with letters. You decide which of them are true and which are false.

P : For every integer $n > 1$, the number $2^n - 1$ is prime.

Q : Every polynomial of degree n has at most n roots.

R : The function $f(x) = x^2$ is continuous.

$S_1: \mathbb{N} \subseteq \emptyset$

$S_2: 0, -1, -2 \cap \mathbb{N} = \emptyset$

Designating statements with letters (as was done above) is a very useful shorthand. In discussing a particular statement, such as “The function $f(x) = x^2$ is continuous,” it is convenient to just refer to it as R to avoid having to write or say it many times.

Statements can contain variables. Here is an example.

P : If an integer x is a multiple of 6, then x is even.

This is a sentence that is true. (All multiples of 6 are even, so no matter which multiple of 6 the integer x happens to be, it is even.) Since the sentence P is definitely true, it is a statement. When a sentence or statement P contains a variable such as x , we sometimes denote it as $P(x)$ to indicate that it is saying something about x . Thus the above statement can be denoted as

$P(x)$: If an integer x is a multiple of 6, then x is even.

A statement or sentence involving two variables might be denoted $P(x, y)$, and so on.

It is quite possible for a sentence containing variables to not be a statement. Consider the following example.

$Q(x)$: The integer x is even.

Is this a statement? Whether it is true or false depends on just which integer x is. It is true if $x = 4$ and false if $x = 7$, etc. But without any stipulations on the value of x it is impossible to say whether $Q(x)$ is true or false. Since it is neither definitely true nor definitely false, $Q(x)$ cannot be a statement. A sentence such as this, whose truth depends on the value of one or more variables, is called an **open sentence**. The variables in an open sentence (or statement) can represent any type of entity, not just numbers. Here is an open sentence where the variables are functions:

$R(f, g)$: The function f is the derivative of the function g .

This open sentence is true if $f(x) = 2x$ and $g(x) = x^2$. It is false if $f(x) = x^3$ and $g(x) = x^2$, etc. We point out that a sentence such as $R(f, g)$ (that involves variables) can be denoted either as $R(f, g)$ or just R . We use the expression $R(f, g)$ when we want to emphasize that the sentence involves variables.

We will have more to say about open sentences later, but for now let's return to statements.

Statements are everywhere in mathematics. Any result or theorem that has been proved true is a statement. The quadratic formula and the Pythagorean theorem are both statements:

P : The solutions of the equation

$$ax^2 + bx + c = 0 \text{ are } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Q : If a right triangle has legs of lengths a and b and hypotenuse of length $a^2 + b^2 = c^2$.

Here is a very famous statement, so famous, in fact, that it has a name. It is called **Fermat's last theorem** after Pierre Fermat, a seventeenth-century French mathematician who scribbled it in the margin of a notebook.

R : For all numbers $a, b, c, n \in \mathbb{N}$ with $n > 2$, it is the case that $a^n + b^n \neq c^n$.

Fermat believed this statement was true. He noted that he could prove it was true, except his notebook's margin was too narrow to contain his proof. It is doubtful that he really had a correct proof in mind, for after his death generations of brilliant mathematicians tried unsuccessfully to prove that his statement was true (or false). Finally, in 1993, Andrew Wiles of Princeton University announced that he had devised a proof. Wiles had worked on the problem for over seven years, and his proof runs through hundreds of pages. The moral of this story is that some true statements are not obviously true.

Here is another statement famous enough to be named. It was first posed in the eighteenth century by the German mathematician Christian Goldbach, and thus is called the Goldbach conjecture:

S : Every even integer greater than 2 is a sum of two prime numbers.

You must agree that S is either true or false. It appears to be true, because when you examine even numbers that are bigger than 2, they seem to be sums of two primes: $4 = 2+2$, $6 = 3+3$, $8 = 3+5$, $10 = 5+5$, $12 = 5+7$, $100 = 17+83$ and so on. But that's not to say there isn't some large even number that's not the sum of two primes. If such a number exists, then S is false. The thing is, in the over 260 years since Goldbach first posed this problem, no one has been able to determine whether it's true or false. But since it is clearly either true or false, S is a statement.

This book is about the methods that can be used to prove that S (or any other statement) is true or false. To prove that a statement is true, we start with obvious statements (or other statements that have been proven true) and use logic to deduce more and more complex statements until finally we obtain a statement such as S . Of course some statements are more difficult to prove than others, and S appears to be notoriously difficult; we will concentrate on statements that are easier to prove.

But the point is this: In proving that statements are true, we use logic to help us understand statements and to combine pieces of information to produce new pieces of information. In the next several sections we explore some standard ways that statements can be combined to form new statements, or broken down into simpler statements.

And, Or, Not

The word “and” can be used to combine two statements to form a new statement. Consider for example the following sentence.

R_1 : The number 2 is even **and** the number 3 is odd.

We recognize this as a true statement, based on our common-sense understanding of the meaning of the word “and.” Notice that R_1 is made up of two simpler statements:

P : The number 2 is even.

Q : The number 3 is odd.

These are joined together by the word “and” to form the more complex statement R_1 . The statement R_1 asserts that P and Q are both true. Since both P and Q are in fact true, the statement R_1 is also true.

Had one or both of P and Q been false, then R_1 would be false. For instance, each of the following statements is false.

R_2 : The number 1 is even **and** the number 3 is odd.

R_3 : The number 2 is even **and** the number 4 is odd.

R_4 : The number 3 is even **and** the number 2 is odd.

From these examples we see that any two statements P and Q can be combined to form a new statement " P **and** Q ." In the spirit of using letters to denote statements, we now introduce the special symbol \wedge to stand for the word "and." Thus if P and Q are statements, $P \wedge Q$ stands for the statement " P **and** Q ." The statement $P \wedge Q$ is true if both P and Q are true; otherwise it is false. This is summarized in the following table, called a **truth table**.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

In this table, T stands for "True," and F stands for "False." (T and F are called **truth values**.) Each line lists one of the four possible combinations or truth values for P and Q , and the column headed by $P \wedge Q$ tells whether the statement $P \wedge Q$ is true or false in each case.

Statements can also be combined using the word "or." Consider the following four statements.

S_1 : The number 2 is even or the number 3 is odd.

S_2 : The number 1 is even or the number 3 is odd.

S_3 : The number 2 is even or the number 4 is odd.

S_4 : The number 3 is even or the number 2 is odd.

In mathematics, the assertion " P **or** Q " is always understood to mean that one or both of P and Q is true. Thus statements S_1 , S_2 , S_3 are all true, while S_4 is false. The symbol \vee is used to stand for the word "or." So if P and Q are statements, $P \vee Q$ represents the statement " P **or** Q ." Here is the truth table.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

It is important to be aware that the meaning of “or” expressed in the above table differs from the way it is sometimes used in everyday conversation. For example, suppose a university official makes the following threat:

You pay your tuition **or** you will be withdrawn from school.

You understand that this means that either you pay your tuition or you will be withdrawn from school, *but not both*. In mathematics we never use the word “or” in such a sense. For us “or” means exactly what is stated in the table for \vee . Thus $P \vee Q$ being true means *one or both* of P and Q is true. If we ever need to express the fact that exactly one of P and Q is true, we use one of the following constructions:

P or Q , but not both.

Either P or Q .

If the university official were a mathematician, he might have qualified his statement in one of the following ways.

Pay your tuition **or** you will be withdrawn from school, **but not both**.

Either you pay your tuition **or** you will be withdrawn from school.

To conclude this section, we mention another way of obtaining new statements from old ones. Given any statement P , we can form the new statement “**It is not true that P .**” For example, consider the following statement.

The number 2 is even.

This statement is true. Now change it by inserting the words “It is not true that” at the beginning:

It is not true that the number 2 is even.

This new statement is false.

For another example, starting with the false statement “ $2 \in \emptyset$ ” we get the true statement “It is not true that $2 \in \emptyset$.”

We use the symbol \sim to stand for the words “It’s not true that,” so $\sim P$ means “**It’s not true that** P .” We often read $\sim P$ simply as “not P .” Unlike \wedge and \vee , which combine two statements, the symbol \sim just alters a single statement. Thus its truth table has just two lines, one for each possible truth value of P .

P	$\sim P$
-----	----------

T	F
---	---

F	T
---	---

The statement $\sim P$ is called the **negation** of P . The negation of a specific statement can be expressed in numerous ways. Consider

P : The number 2 is even.

Here are several ways of expressing its negation.

$\sim P$: It’s not true that the number 2 is even.

$\sim P$: It is false that the number 2 is even.

$\sim P$: The number 2 is not even.

In this section we’ve learned how to combine or modify statements with the operations \wedge , \vee and \sim . Of course we can also apply these operations to open sentences or a mixture of open sentences and statements. For example, $(x \text{ is an even integer}) \wedge (3 \text{ is an odd integer})$ is an open sentence that is a combination of an open sentence and a statement.

Conditional Statements

There is yet another way to combine two statements. Suppose we

have in mind a specific integer a . Consider the following statement about a .

R : If the integer a is a multiple of 6, then a is divisible by 2.

We immediately spot this as a true statement based on our knowledge of integers and the meanings of the words “if” and “then.” If integer a is a multiple of 6, then a is even, so therefore a is divisible by 2. Notice that R is built up from two simpler statements:

P : The integer a is a multiple of 6.

Q : The integer a is divisible by 2.

R : If P, then Q.

In general, given any two statements P and Q whatsoever, we can form the new statement “If P, then Q.” This is written symbolically as $P \Rightarrow Q$ which we read as “If P, then Q,” or “P implies Q.” Like \wedge and \vee , the symbol \Rightarrow has a very specific meaning. When we assert that the statement $P \Rightarrow Q$ is true, we mean that if P is true *then* Q must also be true. (In other words we mean that the condition P being true forces Q to be true.) A statement of form $P \Rightarrow Q$ is called a **conditional** statement because it means Q will be true *under the condition* that P is true.

You can think of $P \Rightarrow Q$ as being a promise that whenever P is true, Q will be true also. There is only one way this promise can be broken (i.e. be false) and that is if P is true but Q is false. Thus the truth table for the promise $P \Rightarrow Q$ is as follows:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Perhaps you are bothered by the fact that $P \Rightarrow Q$ is true in the last two lines of this table. Here’s an example to convince you that the table is correct. Suppose your professor makes the following promise:

If you pass the final exam, **then** you will pass the course.

Your professor is making the promise

(You pass the exam) \Rightarrow (You pass the course).

Under what circumstances did she lie? There are four possible scenarios, depending on whether or not you passed the exam and whether or not you passed the course. These scenarios are tallied in the following table.

You pass exam	You pass course	(You pass exam) \Rightarrow (You pass course)
T	T	T
T	F	F
F	T	T
F	F	T

The first line describes the scenario where you pass the exam and you pass the course. Clearly the professor kept her promise, so we put a T in the third column to indicate that she told the truth. In the second line, you passed the exam, but your professor gave you a failing grade in the course. In this case she broke her promise, and the F in the third column indicates that what she said was untrue.

Now consider the third row. In this scenario you failed the exam but still passed the course. How could that happen? Maybe your professor felt sorry for you. But that doesn't make her a liar. Her only promise was that if you passed the exam then you would pass the course. She did not say passing the exam was the **only way** to pass the course. Since she didn't lie, then she told the truth, so there is a T in the third column.

Finally look at the fourth row. In that scenario you failed the exam and you failed the course. Your professor did not lie; she did exactly what she said she would do. Hence the T in the third column.

In mathematics, whenever we encounter the construction "If P, then Q" it means exactly what the truth table for \Rightarrow expresses. But of course there are other grammatical constructions that also mean $P \Rightarrow Q$. Here is a summary of the main ones.

$$P \Rightarrow Q \left\{ \begin{array}{l} \text{If } P, \text{ then } Q. \\ Q \text{ if } P. \\ Q \text{ whenever } P. \\ Q, \text{ provided that } P. \\ \text{Whenever } P, \text{ then also } Q. \\ P \text{ is a sufficient condition for } Q. \\ \text{For } Q, \text{ it is sufficient that } P. \\ Q \text{ is a necessary condition for } P. \\ \text{For } P, \text{ it is necessary that } Q. \\ P \text{ only if } Q. \end{array} \right.$$

These can all be used in the place of (and mean exactly the same thing as) “If P , then Q .” You should analyze the meaning of each one and convince yourself that it captures the meaning of $P \Rightarrow Q$. For example, $P \Rightarrow Q$ means the condition of P being true is enough (i.e., sufficient) to make Q true; hence “ P is a *sufficient condition* for Q .”

The wording can be tricky. Often an everyday situation involving a conditional statement can help clarify it. For example, consider your professor’s promise:

(You pass the exam) \Rightarrow (You pass the course)

This means that your passing the exam is a sufficient (though perhaps not necessary) condition for your passing the course. Thus your professor might just as well have phrased her promise in one of the following ways.

Passing the exam is a sufficient condition for passing the course.

For you to pass the course, it is sufficient that you pass the exam.

However, when we want to say “If P , then Q ” in everyday conversation, we do not normally express this as “ Q is a *necessary condition* for P ” or “ P only if Q .” But such constructions are not uncommon in mathematics. To understand why they make sense, notice that $P \Rightarrow Q$ being true means that it’s impossible that P is true but Q is false, so in order for P to be true it is necessary that Q is

true; hence “Q is a *necessary condition* for P.” And this means that P can only be true if Q is true, i.e., “P *only if* Q.”

Biconditional Statements

It is important to understand that $P \Rightarrow Q$ is not the same as $Q \Rightarrow P$. To see why, suppose that a is some integer and consider the statements

(a is a multiple of 6) \Rightarrow (a is divisible by 2),

(a is divisible by 2) \Rightarrow (a is a multiple of 6).

The first statement asserts that if a is a multiple of 6 then a is divisible by 2. This is clearly true, for any multiple of 6 is even and therefore divisible by 2. The second statement asserts that if a is divisible by 2 then it is a multiple of 6. This is not necessarily true, for $a = 4$ (for instance) is divisible by 2, yet not a multiple of 6. Therefore the meanings of $P \Rightarrow Q$ and $Q \Rightarrow P$ are in general quite different. The conditional statement $Q \Rightarrow P$ is called the **converse** of $P \Rightarrow Q$, so a conditional statement and its converse express entirely different things.

However, the **contrapositive** of $P \Rightarrow Q$, $\sim Q \Rightarrow \sim P$, is equivalent to $P \Rightarrow Q$. Similarly, the **inverse** of $P \Rightarrow Q$, which is $\sim P \Rightarrow \sim Q$, is equivalent to the converse $Q \Rightarrow P$. In “Truth Tables for Statements,” we will learn how to show these equivalences using a truth table.

But sometimes, if P and Q are just the right statements, it can happen that $P \Rightarrow Q$ and $Q \Rightarrow P$ are both necessarily true. For example, consider the statements

$$(a \text{ is even}) \Rightarrow (a \text{ is divisible by } 2),$$

$$(a \text{ is divisible by } 2) \Rightarrow (a \text{ is even}).$$

No matter what value a has, both of these statements are true. Since both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true, it follows that $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is true.

We now introduce a new symbol \Leftrightarrow to express the meaning of the statement $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. The expression $P \Leftrightarrow Q$ is understood to have exactly the same meaning as $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. According to the previous section, $Q \Rightarrow P$ is read as “ P if Q ,” and $P \Rightarrow Q$ can be read as “ P only if Q .” Therefore we pronounce $P \Leftrightarrow Q$ as “ P if and only if Q .” For example, given an integer a , we have the true statement

$$(a \text{ is even}) \Leftrightarrow (a \text{ is divisible by } 2),$$

which we can read as “*Integer a is even if and only if a is divisible by 2.*”

The truth table for \Leftrightarrow is shown below. Notice that in the first and last rows, both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true (according to the truth table for \Rightarrow), so $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is true, and hence $P \Leftrightarrow Q$ is true. However, in the middle two rows one of $P \Rightarrow Q$ or $Q \Rightarrow P$ is false, so $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is false, making $P \Leftrightarrow Q$ false.

P	Q	P \Leftrightarrow Q
T	T	T
T	F	F
F	T	F
F	F	T

Compare the statement $R : (a \text{ is even}) \Leftrightarrow (a \text{ is divisible by } 2)$ with this truth table. If a is even then the two statements on either side of \Leftrightarrow are true, so according to the table R is true. If a is odd then the two statements on either side of \Leftrightarrow are false, and again according to the table R is true. Thus R is true no matter what value a has. In general, $P \Leftrightarrow Q$ being true means P and Q are both true or both false.

Not surprisingly, there are many ways of saying $P \Leftrightarrow Q$ in English. The following constructions all mean $P \Leftrightarrow Q$:

$$P \Leftrightarrow Q \begin{cases} P \text{ if and only if } Q. \\ P \text{ is a necessary and sufficient condition for } Q. \\ \text{For } P \text{ it is necessary and sufficient that } Q. \\ \text{If } P, \text{ then } Q, \text{ and conversely.} \end{cases}$$

The first three of these just combine constructions from the previous section to express that $P \Rightarrow Q$ and $Q \Rightarrow P$. In the last one, the words “*and conversely*” mean that in addition to “If P , then Q ” being true, the converse statement “If Q , then P ” is also true.

Truth Tables for Statements

You should now know the truth tables for \wedge , \vee , \sim , \Rightarrow and \Leftrightarrow . They should be *internalized* as well as memorized. You must understand the symbols thoroughly, for we now combine them to form more complex statements.

For example, suppose we want to convey that one or the other of P and Q is true but they are not both true. No single symbol expresses this, but we could combine them as

$$(P \vee Q) \wedge \sim (P \wedge Q),$$

which literally means:

P or Q is true, and it is not the case that both P and Q are true.

This statement will be true or false depending on the truth values of P and Q . In fact we can make a truth table for the entire statement. Begin as usual by listing the possible true/false combinations of P and Q on four lines. The statement $(P \vee Q) \wedge \sim (P \wedge Q)$ contains the individual statements $(P \vee Q)$ and $(P \wedge Q)$, so we next tally their truth values in the third and fourth columns. The fifth column lists values for $\sim (P \wedge Q)$, and these are just the opposites of the corresponding entries in the fourth column. Finally, combining

the third and fifth columns with \wedge , we get the values for $(P \vee Q) \wedge \neg(P \wedge Q)$ in the sixth column.

P	Q	$(P \vee Q)$	$(P \wedge Q)$	$\neg(P \wedge Q)$	$(P \vee Q) \wedge \neg(P \wedge Q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

This truth table tells us that $(P \vee Q) \wedge \neg(P \wedge Q)$ is true precisely when one but not both of P and Q are true, so it has the meaning we intended. (Notice that the middle three columns of our truth table are just “helper columns” and are not necessary parts of the table. In writing truth tables, you may choose to omit such columns if you are confident about your work.)

For another example, consider the following familiar statement concerning two real numbers x and y :

The product xy equals zero if and only if $x = 0$ or $y = 0$.

This can be modeled as $(xy = 0) \Leftrightarrow (x = 0 \vee y = 0)$. If we introduce letters P , Q , and R for the statements $xy = 0$, $x = 0$ and $y = 0$, it becomes $P \Leftrightarrow (Q \vee R)$. Notice that the parentheses are necessary here, for without them we wouldn’t know whether to read the statement as $P \Leftrightarrow (Q \vee R)$ or $(P \Leftrightarrow Q) \vee R$.

Making a truth table for $P \Leftrightarrow (Q \vee R)$ entails a line for each T/F combination for the three statements P , Q , and R . The eight possible combinations are tallied in the first three columns of the following table.

P	Q	R	$Q \vee R$	$P \Leftrightarrow (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

We fill in the fourth column using our knowledge of the truth table for \vee . Finally the fifth column is filled in by combining the first and fourth columns with our understanding of the truth table for \Leftrightarrow . The resulting table gives the true/false values of $P \Leftrightarrow (Q \vee R)$ for all values of P , Q , and R .

Notice that when we plug in various values for x and y , the statements $P : xy = 0$, $Q : x = 0$ and $R : y = 0$ have various truth values, but the statement $P \Leftrightarrow (Q \vee R)$ is always true. For example, if $x = 2$ and $y = 3$, then P , Q , and R are all false. This scenario is described in the last row of the table, and there we see that $P \Leftrightarrow (Q \vee R)$ is true. Likewise if $x = 0$ and $y = 7$, then P and Q are true and R is false, a scenario described in the second line of the table, where again $P \Leftrightarrow (Q \vee R)$ is true. There is a simple reason why $P \Leftrightarrow (Q \vee R)$ is true for any values of x and y : It is that $P \Leftrightarrow (Q \vee R)$ represents $(xy = 0) \Leftrightarrow (x = 0 \vee y = 0)$, which is a *true mathematical statement*. It is absolutely impossible for it to be false.

This may make you wonder about the lines in the table where $P \Leftrightarrow (Q \vee R)$ is false. Why are they there? The reason is that $P \Leftrightarrow (Q \vee R)$ can also represent a false statement. To see how, imagine that at the end of the semester your professor makes the following promise.

You pass the class if and only if you get an “A” on the final or you get a “B” on the final.

This promise has the form $P \Leftrightarrow (Q \vee R)$, so its truth values are

tabulated in the above table. Imagine it turned out that you got an “A” on the exam but failed the course. Then surely your professor lied to you. In fact, P is false, Q is true and R is false. This scenario is reflected in the sixth line of the table, and indeed $P \Leftrightarrow (Q \vee R)$ is false (i.e., it is a lie).

The moral of this example is that people can lie, but true mathematical statements **never** lie.

We close this section with a word about the use of parentheses. The symbol \sim is analogous to the minus sign in algebra. It negates the expression it precedes. Thus $\sim P \vee Q$ means $(\sim P) \vee Q$, not $\sim(P \vee Q)$. In $\sim(P \vee Q)$, the value of the entire expression $P \vee Q$ is negated.

Logical Equivalence

In contemplating the truth table for $P \Leftrightarrow Q$, you probably noticed that $P \Leftrightarrow Q$ is true exactly when P and Q are both true or both false. In other words, $P \Leftrightarrow Q$ is true precisely when at least one of the statements $P \wedge Q$ or $\sim P \wedge \sim Q$ is true. This may tempt us to say that $P \Leftrightarrow Q$ means the same thing as $(P \wedge Q) \vee (\sim P \wedge \sim Q)$.

To see if this is really so, we can write truth tables for $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\sim P \wedge \sim Q)$. In doing this, it is more efficient to put these two statements into the same table, as follows. (This table has helper columns for the intermediate expressions $\sim P$, $\sim Q$, $(P \wedge Q)$, and $(\sim P \wedge \sim Q)$.)

P	Q	$\sim P$	$\sim Q$	$(P \wedge Q)$	$(\sim P \wedge \sim Q)$	$(P \wedge Q) \vee (\sim P \wedge \sim Q)$	$P \Leftrightarrow Q$
T	T	F	F	T	F	T	T
T	F	F	T	F	F	F	F
F	T	T	F	F	F	F	F
F	F	T	T	F	T	T	T

The table shows that $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\sim P \wedge \sim Q)$ have the same truth value, no matter the values P and Q . It is as if $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\sim P \wedge \sim Q)$ are algebraic expressions that are equal no matter what is “plugged into” variables P and Q . We express this state of affairs by writing

$$P \Leftrightarrow Q = (P \wedge Q) \vee (\sim P \wedge \sim Q)$$

and saying that $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\sim P \wedge \sim Q)$ are **logically equivalent**.

In general, two statements are **logically equivalent** if their truth values match up line-for-line in a truth table.

Logical equivalence is important because it can give us different (and potentially useful) ways of looking at the same thing. As an example, the following table shows that $P \Rightarrow Q$ is logically equivalent to $(\sim Q) \Rightarrow (\sim P)$.

P	Q	$\sim P$	$\sim Q$	$(\sim Q) \Rightarrow (\sim P)$	$P \Rightarrow Q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The fact that $P \Rightarrow Q = (\sim Q) \Rightarrow (\sim P)$ is useful because so many theorems have the form $P \Rightarrow Q$. As we will see in Chapter 5, proving such a theorem may be easier if we express it in the logically equivalent form $(\sim Q) \Rightarrow (\sim P)$.

There are two pairs of logically equivalent statements that come up again and again throughout this book and beyond. They are prevalent enough to be dignified by a special name: **DeMorgan's laws**.

Fact: DeMorgan's Laws

1. $\neg(P \wedge Q) = (\neg P) \vee (\neg Q)$
2. $\neg(P \vee Q) = (\neg P) \wedge (\neg Q)$

The first of DeMorgan's laws is verified by the following table. You are asked to verify the second in one of the exercises.

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

DeMorgan's laws are actually very natural and intuitive. Consider the statement $\neg(P \wedge Q)$, which we can interpret as meaning that it is not the case that both P and Q are true. If it is not the case that both P and Q are true, then at least one of P or Q is false, in which case $(\neg P) \vee (\neg Q)$ is true. Thus $\neg(P \wedge Q)$ means the same thing as $(\neg P) \vee (\neg Q)$.

DeMorgan's laws can be very useful. Suppose we happen to know that some statement having form $\neg(P \vee Q)$ is true. The second of DeMorgan's laws tells us that $(\neg Q) \wedge (\neg P)$ is also true, hence $\neg P$ and $\neg Q$ are both true as well. Being able to quickly obtain such additional pieces of information can be extremely useful.

Here is a summary of some significant logical equivalences. Those that are not immediately obvious can be verified with a truth table.

Contrapositive law $\{ P \Rightarrow Q = (\sim Q) \Rightarrow (\sim P) \}$

$$\text{DeMorgan's laws} \begin{cases} \sim (P \wedge Q) = \sim P \vee \sim Q \\ \sim (P \vee Q) = \sim P \wedge \sim Q \end{cases}$$

$$\text{Commutative laws} \begin{cases} (P \wedge Q) = P \wedge Q \\ (P \vee Q) = P \vee Q \end{cases}$$

$$\text{Distributive laws} \begin{cases} P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R) \\ P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R) \end{cases}$$

$$\text{Associative laws} \begin{cases} P \wedge (Q \wedge R) = (P \wedge Q) \wedge R \\ P \vee (Q \vee R) = (P \vee Q) \vee R \end{cases}$$

Notice how the distributive law $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$ has the same structure as the distributive law $p(q + r) = p \cdot q + p \cdot r$ from algebra. Concerning the associative laws, the fact that $P \wedge (Q \wedge R) = (P \wedge Q) \wedge R$ means that the position of the parentheses is irrelevant, and we can write this as $P \wedge Q \wedge R$ without ambiguity. Similarly, we may drop the parentheses in an expression such as $P \vee (Q \vee R)$.

But parentheses are essential when there is a mix of \wedge and \vee , as in $P \vee (Q \wedge R)$. Indeed, $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge R$ are **not** logically equivalent.

Negating Statements

Given a statement R , the statement $\sim R$ is called the **negation** of R . If R is a complex statement, then it is often the case that its negation $\sim R$ can be written in a simpler or more useful form. The process of finding this form is called **negating** R . In proving theorems it is often necessary to negate certain statements. We now investigate how to do this.

We have already examined part of this topic. **DeMorgan's laws**

$$\sim (P \wedge Q) = (\sim P) \vee (\sim Q)$$

$$\sim (P \vee Q) = (\sim P) \wedge (\sim Q)$$

(from "Logical Equivalence") can be viewed as rules that tell us

how to negate the statements $P \wedge Q$ and $P \vee Q$. Here are some examples that illustrate how DeMorgan's laws are used to negate statements involving "and" or "or."

Example 5

Consider negating the following statement.

R : You can solve it by factoring or with the quadratic formula.

Now, R means (You can solve it by factoring) \vee (You can solve it with Q.F.), which we will denote as $P \vee Q$. The negation of this is

$$\sim(P \vee Q) = (\sim P) \wedge (\sim Q).$$

Therefore, in words, the negation of R is

$\sim R$: You can't solve it by factoring and you can't solve it with the quadratic formula.

Maybe you can find $\sim R$ without invoking DeMorgan's laws. That is good; you have internalized DeMorgan's laws and are using them unconsciously.

Example 6

We will negate the following sentence.

R : The numbers x and y are both odd.

This statement means $(x \text{ is odd}) \wedge (y \text{ is odd})$, so its negation is

$$\begin{aligned}\sim[(x \text{ is odd}) \wedge (y \text{ is odd})] &= \sim(x \text{ is odd}) \vee \sim(y \text{ is odd}) \\ (x \text{ is odd}) \wedge (y \text{ is odd}) &= (x \text{ is even}) \vee (y \text{ is even}).\end{aligned}$$

Therefore the negation of R can be expressed in the following ways:

$$\begin{aligned}\sim R &: \text{The number } x \text{ is even or the number } y \text{ is even.} \\ \sim R &: \text{At least one of } x \text{ and } y \text{ is even.}\end{aligned}$$

Now let's move on to a slightly different kind of problem. It's often necessary to find the negations of quantified statements. For example, consider $\sim(\forall x \in \mathbb{N}, P(x))$. Reading this in words, we have the following:

It is not the case that $P(x)$ is true for all natural numbers x .

This means $P(x)$ is false for at least one x . In symbols, this is $\exists x \in \mathbb{N}, \sim P(x)$. Thus $\sim(\forall x \in \mathbb{N}, P(x)) = \exists x \in \mathbb{N}, \sim P(x)$. Similarly, you can reason out that $\sim(\exists x \in \mathbb{N}, P(x)) = \forall x \in \mathbb{N}, \sim P(x)$. In general:

$$\begin{aligned}\sim(\forall x \in S, P(x)) &= \exists x \in S, \sim P(x) \\ \sim(\exists x \in S, P(x)) &= \forall x \in S, \sim P(x)\end{aligned}$$

Logical Inference

Suppose we know that a statement of form $P \Rightarrow Q$ is true. This tells us that whenever P is true, Q will also be true. By itself, $P \Rightarrow Q$ being true does not tell us that either P or Q is true (they could both be false, or P could be false and Q true). However if in addition we happen to know that P is true then it must be that Q is true. This is called a **logical inference**: Given two true statements we can infer that a third statement is true. In this instance true statements

$P \Rightarrow Q$ and P are “added together” to get Q . This is described below with $P \Rightarrow Q$ and P stacked one atop the other with a line separating them from Q . The intended meaning is that $P \Rightarrow Q$ combined with P produces Q .

$$\frac{P \Rightarrow Q}{P} Q$$

$$\frac{P \Rightarrow Q}{\sim Q} \sim P$$

$$\frac{P \vee Q}{\sim P} Q$$

Two other logical inferences are listed above. In each case you should convince yourself (based on your knowledge of the relevant truth tables) that the truth of the statements above the line forces the statement below the line to be true.

Following are some additional useful logical inferences. The first expresses the obvious fact that if P and Q are both true then the statement $P \wedge Q$ will be true. On the other hand, $P \wedge Q$ being true forces P (also Q) to be true. Finally, if P is true, then $P \vee Q$ must be true, no matter what statement Q is.

$$\frac{P}{Q} P \wedge Q$$

$$\frac{P \wedge Q}{P}$$

$$\frac{P}{P \vee Q}$$

These inferences are so intuitively obvious that they scarcely need to be mentioned. However, they represent certain patterns of reasoning that we will frequently apply to sentences in proofs, so we should be cognizant of the fact that we are using them.

The first two statements in each case are called “premises” and the final statement is the “conclusion.” We combine premises with \wedge (“and”). The premises together imply the conclusion. Thus, the first argument would have $((P \Rightarrow Q) \wedge P) \Rightarrow Q$ as its symbolic statement.

An Important Note

It is important to be aware of the reasons that we study logic. There are three very significant reasons. First, the truth tables we studied tell us the exact meanings of the words such as “and,” “or,” “not,” and so on. For instance, whenever we use or read the “If..., then” construction in a mathematical context, logic tells us exactly what is meant. Second, the rules of inference provide a system in which we can produce new information (statements) from known information. Finally, logical rules such as DeMorgan’s laws help us correctly change certain statements into (potentially more useful) statements with the same meaning. Thus logic helps us understand the meanings of statements and it also produces new meaningful statements.

Logic is the glue that holds strings of statements together and pins down the exact meaning of certain key phrases such as the “If..., then” or “For all” constructions. Logic is the common language that all mathematicians use, so we must have a firm grip on it in order to write and understand mathematics.

But despite its fundamental role, logic’s place is in the background of what we do, not the forefront. From here on, the beautiful symbols \wedge , \vee , \Rightarrow , \Leftrightarrow , \neg , \forall and \exists are rarely written. But we are aware of their meanings constantly. When reading or writing a sentence involving mathematics we parse it with these symbols, either mentally or on scratch paper, so as to understand the true and unambiguous meaning.

3. Truth Tables and Analyzing Arguments: Examples

Truth Tables

Because complex Boolean statements can get tricky to think about, we can create a **truth table** to keep track of what truth values for the simple statements make the complex statement true and false

Truth Table

A table showing what the resulting truth value of a complex statement is for all the possible truth values for the simple statements.

Example 1

Suppose you're picking out a new couch, and your significant other says "get a sectional or something with a chaise."

This is a complex statement made of two simpler

conditions: “is a sectional,” and “has a chaise.” For simplicity, let’s use S to designate “is a sectional,” and C to designate “has a chaise.” The condition S is true if the couch is a sectional.

A truth table for this would look like this:

S	C	S or C
T	T	T
T	F	T
F	T	T
F	F	F

In the table, T is used for true, and F for false. In the first row, if S is true and C is also true, then the complex statement “S or C” is true. This would be a sectional that also has a chaise, which meets our desire.

Remember also that *or* in logic is not exclusive; if the couch has both features, it does meet the condition.

To shorthand our notation further, we’re going to introduce some symbols that are commonly used for *and*, *or*, and *not*.

Symbols

The symbol \wedge is used for *and*: A and B is notated $A \wedge B$.

The symbol \vee is used for or: A or B is notated $A \vee B$

The symbol \sim is used for not: not A is notated $\sim A$

You can remember the first two symbols by relating them to the shapes for the union and intersection. $A \wedge B$ would be the elements that exist in both sets, in $A \cap B$. Likewise, $A \vee B$ would be the elements that exist in either set, in $A \cup B$.

In the previous example, the truth table was really just summarizing what we already know about how the or statement work. The truth tables for the basic *and*, *or*, and *not* statements are shown below.

Basic Truth Tables

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

A	$\sim A$
T	F
F	T

Truth tables really become useful when analyzing more complex Boolean statements.

Example 2

Create a truth table for the statement $A \wedge \sim(B \vee C)$

It helps to work from the inside out when creating truth tables, and create tables for intermediate operations. We start by listing all the possible truth value combinations for A, B, and C. Notice how the first column contains 4 Ts followed by 4 Fs, the second column contains 2 Ts, 2 Fs, then repeats, and the last column alternates. This pattern ensures that all combinations are considered. Along with

those initial values, we'll list the truth values for the innermost expression, $B \vee C$.

A	B	C	$B \vee C$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Next we can find the negation of $B \vee C$, working off the $B \vee C$ column we just created.

A	B	C	$B \vee C$	$\sim(B \vee C)$
T	T	T	T	F
T	T	F	T	F
T	F	T	T	F
T	F	F	F	T
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

Finally, we find the values of A and $\sim(B \vee C)$

A	B	C	$B \vee C$	$\sim(B \vee C)$	$A \wedge \sim(B \vee C)$
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	F	T	F

It turns out that this complex expression is only true in one case: if A is true, B is false, and C is false.

When we discussed conditions earlier, we discussed the type where we take an action based on the value of the condition. We are now going to talk about a more general version of a conditional, sometimes called an **implication**.

Implications

Implications are logical conditional sentences stating that a statement p , called the antecedent, implies a consequence q .

Implications are commonly written as $p \rightarrow q$

Implications are similar to the conditional statements we looked at earlier; $p \rightarrow q$ is typically written as “if p then q ,” or “ p therefore q .” The difference between implications and conditionals is that conditionals we discussed earlier suggest an action—if the condition is true, then we take some action as a result. Implications are a logical statement that suggest that the consequence must logically follow if the antecedent is true.

Example 3

The English statement “If it is raining, then there are clouds in the sky” is a logical implication. It is a valid argument because if the antecedent “it is raining” is true, then the consequence “there are clouds in the sky” must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining. If the antecedent is false, then the implication becomes irrelevant.

Example 4

A friend tells you that “if you upload that picture to Facebook, you’ll lose your job.” There are four possible outcomes:

1. You upload the picture and keep your job
2. You upload the picture and lose your job

3. You don't upload the picture and keep your job
4. You don't upload the picture and lose your job

There is only one possible case where your friend was lying—the first option where you upload the picture and keep your job. In the last two cases, your friend didn't say anything about what would happen if you didn't upload the picture, so you can't conclude their statement is invalid, even if you didn't upload the picture and still lost your job.

In traditional logic, an implication is considered valid (true) as long as there are no cases in which the antecedent is true and the consequence is false. It is important to keep in mind that symbolic logic cannot capture all the intricacies of the English language.

Truth Values for Implications

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 5

Construct a truth table for the statement $(m \wedge \sim p) \rightarrow r$

We start by constructing a truth table for the antecedent.

m	p	$\sim p$	$m \wedge \sim p$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Now we can build the truth table for the implication

m	p	$\sim p$	$m \wedge \sim p$	r	$(m \wedge \sim p) \rightarrow r$
T	T	F	F	T	T
T	F	T	T	T	T
F	T	F	F	T	T
F	F	T	F	T	T
T	T	F	F	F	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	F	F	T

In this case, when m is true, p is false, and r is false, then the antecedent $m \wedge \sim p$ will be true but the consequence false, resulting in an invalid implication; every other case gives a valid implication.

For any implication, there are three related statements, the converse, the inverse, and the contrapositive.

Related Statements

The original implication is “if p then q ”: $p \rightarrow q$

The converse is “if q then p ”: $q \rightarrow p$

The inverse is “if not p then not q ”: $\sim p \rightarrow \sim q$

The contrapositive is “if not q then not p ”: $\sim q \rightarrow \sim p$

Example 6

Consider again the valid implication “If it is raining, then there are clouds in the sky.”

The converse would be “If there are clouds in the sky, it is raining.” This is certainly not always true.

The inverse would be “If it is not raining, then there are not clouds in the sky.” Likewise, this is not always true.

The contrapositive would be “If there are not clouds in the sky, then it is not raining.” This statement is valid, and is equivalent to the original implication.

Looking at truth tables, we can see that the original conditional and

the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.

		Implication	Converse	Inverse	Contrapositive
p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Equivalence

A conditional statement and its contrapositive are logically equivalent.

The converse and inverse of a statement are logically equivalent.

Arguments

A logical argument is a claim that a set of premises support a conclusion. There are two general types of arguments: inductive and deductive arguments.

Argument types

An **inductive** argument uses a collection of specific examples as its premises and uses them to propose a general conclusion.

A **deductive** argument uses a collection of general statements as its premises and uses them to propose a specific situation as the conclusion.

Example 7

The argument “when I went to the store last week I forgot my purse, and when I went today I forgot my purse. I always forget my purse when I go the store” is an inductive argument.

The premises are:

I forgot my purse last week

I forgot my purse today

The conclusion is:

I always forget my purse

Notice that the premises are specific situations, while the conclusion is a general statement. In this case, this is a

fairly weak argument, since it is based on only two instances.

Example 8

The argument “every day for the past year, a plane flies over my house at 2pm. A plane will fly over my house every day at 2pm” is a stronger inductive argument, since it is based on a larger set of evidence.

Evaluating inductive arguments

An inductive argument is never able to prove the conclusion true, but it can provide either weak or strong evidence to suggest it may be true.

Many scientific theories, such as the big bang theory, can never be proven. Instead, they are inductive arguments supported by a wide variety of evidence. Usually in science, an idea is considered a hypothesis until it has been well tested, at which point it graduates to being considered a theory. The commonly known scientific theories, like Newton’s theory of gravity, have all stood up to years

of testing and evidence, though sometimes they need to be adjusted based on new evidence. For gravity, this happened when Einstein proposed the theory of general relativity.

A deductive argument is more clearly valid or not, which makes them easier to evaluate.

Evaluating deductive arguments

A deductive argument is considered valid if all the premises are true, and the conclusion follows logically from those premises. In other words, the premises are true, and the conclusion follows necessarily from those premises.

Example 9

The argument “All cats are mammals and a tiger is a cat, so a tiger is a mammal” is a valid deductive argument.

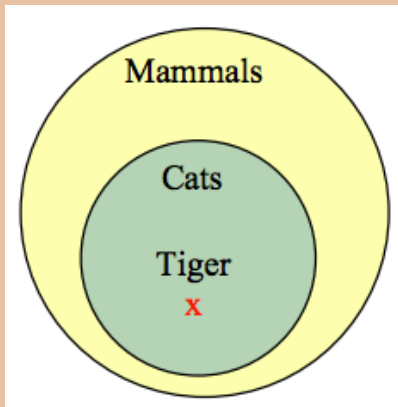
The premises are:

All cats are mammals

A tiger is a cat

The conclusion is:

A tiger is a mammal



Both the premises are true. To see that the premises must logically lead to the conclusion, one approach would be use a Venn diagram. From the first premise, we can conclude that the set of cats is a subset of the set of mammals. From the second premise, we are told that a tiger lies within the set of cats. From that, we can see in the Venn diagram that the tiger also lies inside the set of mammals, so the conclusion is valid.

Analyzing Arguments with Venn Diagrams¹

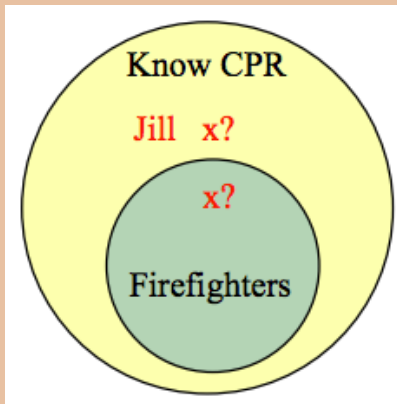
To analyze an argument with a Venn diagram

1. Draw a Venn diagram based on the premises of the argument
2. If the premises are insufficient to determine what determine the location of an element, indicate that.
3. The argument is valid if it is clear that the conclusion must be true

Example 10

Premise:	All firefighters know CPR
Premise:	Jill knows CPR
Conclusion:	Jill is a firefighter

1. Technically, these are Euler circles or Euler diagrams, not Venn diagrams, but for the sake of simplicity we'll continue to call them Venn diagrams.



From the first premise, we know that firefighters all lie inside the set of those who know CPR. From the second premise, we know that Jill is a member of that larger set, but we do not have enough information to know if she also is a member of the smaller subset that is firefighters.

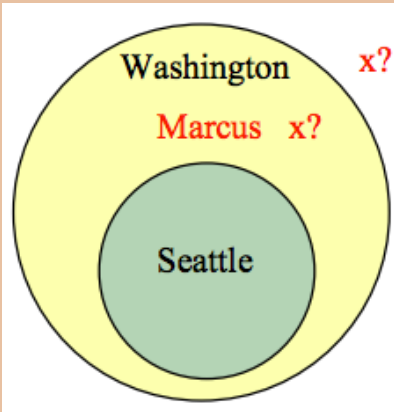
Since the conclusion does not necessarily follow from the premises, this is an invalid argument, regardless of whether Jill actually is a firefighter.

It is important to note that whether or not Jill is actually a firefighter is not important in evaluating the validity of the argument; we are only concerned with whether the premises are enough to prove the conclusion.

In addition to these categorical style premises of the form “all _____,” “some _____,” and “no _____,” it is also common to see premises that are implications.

Example 11

Premise: If you live in Seattle, you live in Washington.
Premise: Marcus does not live in Seattle
Conclusion: Marcus does not live in Washington



From the first premise, we know that the set of people who live in Seattle is inside the set of those who live in Washington. From the second premise, we know that Marcus does not lie in the Seattle set, but we have insufficient information to know whether or not Marcus lives in Washington or not. This is an invalid argument.

Example 12

Consider the argument “You are a married man, so you must have a wife.”

This is an invalid argument, since there are, at least in parts of the world, men who are married to other men, so the premise not insufficient to imply the conclusion.

Some arguments are better analyzed using truth tables.

Example 13

Consider the argument:

Premise: If you bought bread, then you went to the store

Premise: You bought bread

Conclusion: You went to the store

While this example is hopefully fairly obviously a valid argument, we can analyze it using a truth table by representing each of the premises symbolically. We can then look at the implication that the premises together imply the conclusion. If the truth table is a tautology (always true), then the argument is valid.

We'll get B represent “you bought bread” and S represent “you went to the store”. Then the argument becomes:

Premise: $B \rightarrow S$

Premise: B

Conclusion: S

To test the validity, we look at whether the combination of both premises implies the conclusion; is it true that $[(B \rightarrow S) \wedge B] \rightarrow S$?

B	S	$B \rightarrow S$	$(B \rightarrow S) \wedge B$	$[(B \rightarrow S) \wedge B] \rightarrow S$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since the truth table for $[(B \rightarrow S) \wedge B] \rightarrow S$ is always true, this is a valid argument.

Analyzing arguments using truth tables

To analyze an argument with a truth table:

1. Represent each of the premises symbolically
2. Create a conditional statement, joining all the premises with and to form the antecedent, and using the conclusion as the consequent.
3. Create a truth table for that statement. If it is always true, then the argument is valid.

Example 14

Premise: If I go to the mall, then I'll buy new jeans

Premise: If I buy new jeans, I'll buy a shirt to go with it

Conclusion: If I got to the mall, I'll buy a shirt.

Let M = I go to the mall, J = I buy jeans, and S = I buy a shirt.

The premises and conclusion can be stated as:

Premise: $M \rightarrow J$

Premise: $J \rightarrow S$

Conclusion: $M \rightarrow S$

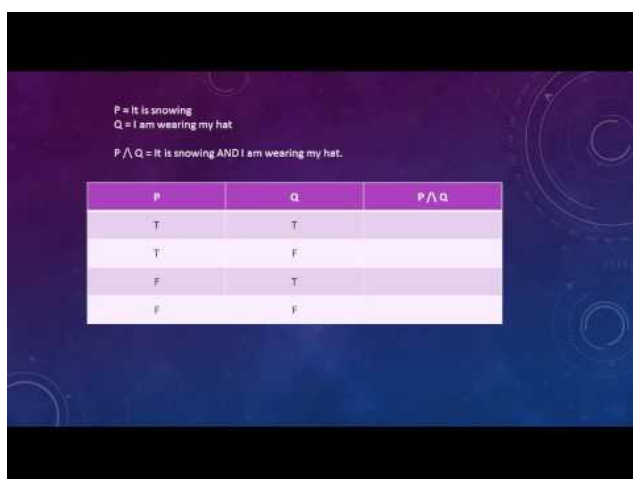
We can construct a truth table for $[(M \rightarrow J) \wedge (J \rightarrow S)] \rightarrow (M \rightarrow S)$

M	J	S	$M \rightarrow J$	$J \rightarrow S$	$(M \rightarrow J) \wedge (J \rightarrow S)$	$M \rightarrow S$	$[(M \rightarrow J) \wedge (J \rightarrow S)] \rightarrow (M \rightarrow S)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the truth table, we can see this is a valid argument.

4. Truth Tables: Conjunction and Disjunction

This video explores the example “It is snowing OR I am wearing my hat,” and “It is snowing AND I am wearing my hat.”



P = It is snowing
Q = I am wearing my hat
 $P \wedge Q$ = It is snowing AND I am wearing my hat.

P	Q	$P \wedge Q$
T	T	
T	F	
F	T	
F	F	

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[https://library.achievingthedream.org/
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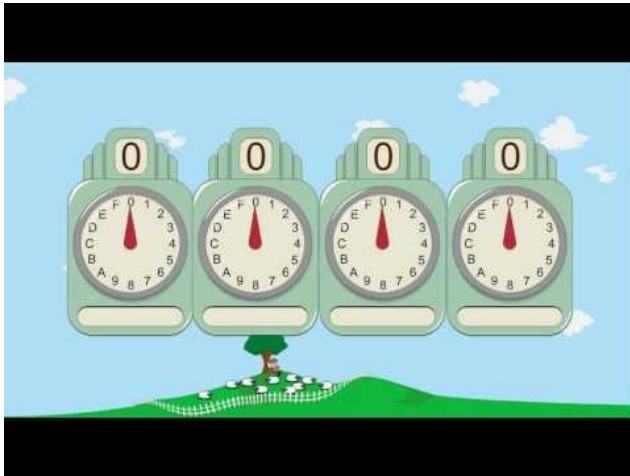
PART IV

MODULE 3: NUMERATION SYSTEM

5. Binary, Octal, and Hexadecimal

In modern computing and digital electronics, the most commonly used bases are decimal (base 10), binary (base 2), octal (base 8), and hexadecimal (base 16). If we are converting between two bases other than decimal, we typically have to convert the number to base 10 first, and then convert that number to the second base. However, we can easily convert directly from binary to octal, and vice versa, and from binary to hexadecimal, and vice versa.

This video gives a basic introduction to these conversions:



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hostosintrocollegemath/?p=25](https://library.achievingthedream.org/hostosintrocollegemath/?p=25)

For another description, this one is more like a math lecture:

Binary ↔ Decimal

Oct ↔ **Hexadecimal**
 $6 + 10 = 16$

msb → 16 ← lsb

0b 011010010110 Binary
↓
0x 6 9 6 Hex
↓
0x 0 A D Hex
↓
Binary

Hex 0 1 2 3 4 5 6 7 8 9 A B C D E F

Binary 8 4 2 1

$2^4 = 16$

0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
0	1	0	1
0	1	1	0
0	1	1	1
1	0	0	0
1	0	0	1
1	0	1	0
1	0	1	1
1	1	0	0
1	1	0	1
1	1	1	0
1	1	1	1

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<https://library.achievingthedream.org/hostosintrocollegemath/?p=25>

For further clarification, recall that the numbers 0 through 7 can be represented by up to three digits in base two. In base eight, these numbers are represented by a single digit.

Base 2 (binary) number	Base 10 (decimal) equivalent	Base 8 (octal) number
000	0	0
001	1	1
010	2	2
011	3	3
100	4	4
101	5	5
110	6	6
111	7	7

Now when we get to the number 8, we need four digits in base 2 and two digits in base 8. In fact, the numbers 8 through 63 can be represented by two digits in base 8. We need four, five, or six digits in base 2 to represent these same numbers:

Base 2 number	Base 10 equivalent	Base 8 number
1000	8	$10 = 1 \times 8 + 0 \times 1$
1001	9	$11 = 1 \times 8 + 1 \times 1$
1010	10	$12 = 1 \times 8 + 2 \times 1$
...
111100	60	$74 = 7 \times 8 + 4 \times 1$
111101	61	$75 = 7 \times 8 + 5 \times 1$
111110	62	$76 = 7 \times 8 + 6 \times 1$
111111	63	$77 = 7 \times 8 + 7 \times 1$

The number 64 in base 8 is represented by $100_8 = 1 \times 8^2 + 0 \times 8^1 + 0 \times 8^0 = 1 \times 64 + 0 \times 8 + 0 \times 1$. In base 2, this would be 1000000_2 . Do you see a pattern here? For a single digit in base 8, we need up to three digits in base 2. For two digits in base 8, we need 4, 5, or 6 digits in base 2. For three digits in base 8, we need 7, 8, or 9 digits in base 2. For each additional digit in base 8, we need up to three

spaces to represent it in base 2. **Here's a way to remember this: $2^3 = 8$, so we need three spaces.**

A couple of examples would help here.

1. Convert the number 6157_8 to base 2. We split each digit in base 8 to three digits in base 2, using the three digit base 2 equivalent, so $6_8 = 110_2$, $1_8 = 001_2$, etc.
2. Convert the number 10111011001010_2 to base 8. Split this number into sets of three, **starting with the right-most digit**, then convert each set of three to its equivalent in base 8.

For hexadecimal (base 16), we need up to four digits in binary to represent each single digit. Remember this by recalling that $2^4 = 16$, so we need four digits.

You may want to print out copies of these worksheets to help you with your conversions between binary and octal or hexadecimal:

- Converting from Binary to Octal
- Converting from Binary to Hexadecimal

If you would like to quiz yourself on converting the numbers 0 through 255 to binary, octal, and hexadecimal (and between those bases), here's a link to the representations of those numbers: [Binary, Octal, and Hexadecimal Numbers](#).

6. Numeration

Historical Counting Systems Introduction and Basic Number and Counting Systems

Introduction

As we begin our journey through the history of mathematics, one question to be asked is “Where do we start?” Depending on how you view mathematics or numbers, you could choose any of a number of launching points from which to begin. Howard Eves suggests the following list of possibilities.¹

Where to start the study of the history of mathematics...

- At the first logical geometric “proofs” traditionally credited to Thales of Miletus (600 BCE).
- With the formulation of methods of measurement made by the Egyptians and Mesopotamians/Babylonians.
- Where prehistoric peoples made efforts to organize the concepts of size, shape, and number.
- In pre-human times in the very simple number sense and pattern recognition that can be displayed by certain animals, birds, etc.
- Even before that in the amazing relationships of numbers and shapes found in plants.
- With the spiral nebulae, the natural course of planets, and

1. Eves, Howard; An Introduction to the History of Mathematics, p. 9.

other universe phenomena.

We can choose no starting point at all and instead agree that mathematics has *always* existed and has simply been waiting in the wings for humans to discover. Each of these positions can be defended to some degree and which one you adopt (if any) largely depends on your philosophical ideas about mathematics and numbers.

Nevertheless, we need a starting point. Without passing judgment on the validity of any of these particular possibilities, we will choose as our starting point the emergence of the idea of number and the process of counting as our launching pad. This is done primarily as a practical matter given the nature of this course. In the following chapter, we will try to focus on two main ideas. The first will be an examination of basic number and counting systems and the symbols that we use for numbers. We will look at our own modern (Western) number system as well those of a couple of selected civilizations to see the differences and diversity that is possible when humans start counting. The second idea we will look at will be base systems. By comparing our own base-ten (decimal) system with other bases, we will quickly become aware that the system that we are so used to, when slightly changed, will challenge our notions about numbers and what symbols for those numbers actually mean.

Recognition of More vs. Less

The idea of number and the process of counting goes back far beyond history began to be recorded. There is some archeological evidence that suggests that humans were counting as far back as 50,000 years ago.² However, we do not really know how this process

2. Eves, p. 9.

started or developed over time. The best we can do is to make a good guess as to how things progressed. It is probably not hard to believe that even the earliest humans had some sense of *more* and *less*. Even some small animals have been shown to have such a sense. For example, one naturalist tells of how he would secretly remove one egg each day from a plover's nest. The mother was diligent in laying an extra egg every day to make up for the missing egg. Some research has shown that hens can be trained to distinguish between even and odd numbers of pieces of food.³ With these sorts of findings in mind, it is not hard to conceive that early humans had (at least) a similar sense of more and less. However, our conjectures about how and when these ideas emerged among humans are simply that; educated guesses based on our own assumptions of what might or could have been.

The Need for Simple Counting

As societies and humankind evolved, simply having a sense of more or less, even or odd, etc., would prove to be insufficient to meet the needs of everyday living. As tribes and groups formed, it became important to be able to know how many members were in the group, and perhaps how many were in the enemy's camp. Certainly it was important for them to know if the flock of sheep or other possessed animals were increasing or decreasing in size. "Just how many of them do we have, anyway?" is a question that we do not have a hard time imagining them asking themselves (or each other).

In order to count items such as animals, it is often conjectured that one of the earliest methods of doing so would be with "tally sticks." These are objects used to track the numbers of items to

3. McLeish, John; *The Story of Numbers—How Mathematics Has Shaped Civilization*, p. 7.

be counted. With this method, each “stick” (or pebble, or whatever counting device being used) represents one animal or object. This method uses the idea of **one to one correspondence**. In a one to one correspondence, items that are being counted are uniquely linked with some counting tool.

In the picture to the right, you see each stick corresponding to one horse. By examining the collection of sticks in hand one knows how many animals should be present. You can imagine the usefulness of such a system, at least for smaller numbers of items to keep track of. If a herder wanted to “count off” his animals to make sure they were all present, he could mentally (or methodically) assign each

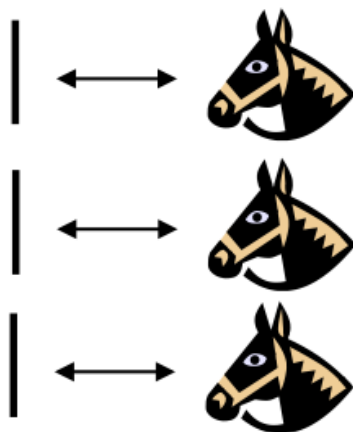


Figure 1.

(or methodically) assign each stick to one animal and continue to do so until he was satisfied that all were accounted for.

Of course, in our modern system, we have replaced the sticks with more abstract objects. In particular, the top stick is replaced with our symbol “1,” the second stick gets replaced by a “2” and the third stick is represented by the symbol “3,” but we are getting ahead of ourselves here. These modern symbols took many centuries to emerge.

Another possible way of employing the “tally stick” counting method is by making marks or cutting notches into pieces of wood, or even tying knots in string (as we shall see later). In 1937, Karl Absalom discovered a wolf bone that goes back possibly 30,000

years. It is believed to be a counting device.⁴ Another example of this kind of tool is the Ishango Bone, discovered in 1960 at Ishango, and shown below.⁵ It is reported to be between six and nine thousand years old and shows what appear to be markings used to do counting of some sort.

The markings on rows (a) and (b) each add up to 60. Row (b) contains the prime numbers between 10 and 20. Row (c) seems to illustrate for the method of doubling and multiplication used by the Egyptians. It is believed that this may also represent a lunar phase counter.

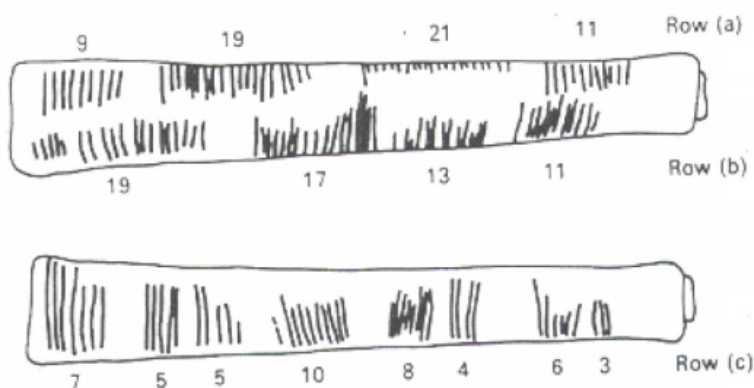


Figure 2.

4. Bunt, Lucas; Jones, Phillip; Bedient, Jack; *The Historical Roots of Elementary Mathematics*, p. 2.

5. http://www.math.buffalo.edu/mad/Ancient-Africa/mad_zaire-uganda.html

Spoken Words

As methods for counting developed, and as language progressed as well, it is natural to expect that spoken words for numbers would appear. Unfortunately, the developments of these words, especially those corresponding to the numbers from one through ten, are not easy to trace. Past ten, however, we do see some patterns:

- Eleven comes from “ein lifon,” meaning “one left over.”
- Twelve comes from “twe lif,” meaning “two left over.”
- Thirteen comes from “Three and ten” as do fourteen through nineteen.
- Twenty appears to come from “twe-tig” which means “two tens.”
- Hundred probably comes from a term meaning “ten times.”

Written Numbers

When we speak of “written” numbers, we have to be careful because this could mean a variety of things. It is important to keep in mind that modern paper is only a little more than 100 years old, so “writing” in times past often took on forms that might look quite unfamiliar to us today.

As we saw earlier, some might consider wooden sticks with notches carved in them as writing as these are means of recording information on a medium that can be “read” by others. Of course, the symbols used (simple notches) certainly did not leave a lot of flexibility for communicating a wide variety of ideas or information.

Other mediums on which “writing” may have taken place include carvings in stone or clay tablets, rag paper made by hand (twelfth century in Europe, but earlier in China), papyrus (invented by the Egyptians and used up until the Greeks), and parchments from animal skins. And these are just a few of the many possibilities.

These are just a few examples of early methods of counting and simple symbols for representing numbers. Extensive books, articles and research have been done on this topic and could provide enough information to fill this entire course if we allowed it to. The range and diversity of creative thought that has been used in the past to describe numbers and to count objects and people is staggering. Unfortunately, we don't have time to examine them all, but it is fun and interesting to look at one system in more detail to see just how ingenious people have been.

The Number and Counting System of the Inca Civilization

Background

There is generally a lack of books and research material concerning the historical foundations of the Americas. Most of the “important” information available concentrates on the eastern hemisphere, with Europe as the central focus. The reasons for this may be twofold: first, it is thought that there was a lack of specialized mathematics in the American regions; second, many of the secrets of ancient mathematics in the Americas have been closely guarded.⁶ The Peruvian system does not seem to be an exception here. Two researchers, Leland Locke and Erland Nordenskiöld, have carried out research that has attempted to discover what mathematical knowledge was known by the Incas and how they used the Peruvian quipu, a counting system using cords and knots, in their

6. Diana, Lind Mae; *The Peruvian Quipu in Mathematics Teacher*, Issue 60 (Oct., 1967), p. 623–28.

mathematics. These researchers have come to certain beliefs about the quipu that we will summarize here.

Counting Boards

It should be noted that the Incas did not have a complicated system of computation. Where other peoples in the regions, such as the Mayans, were doing computations related to their rituals and calendars, the Incas seem to have been more concerned with the simpler task of record-keeping. To do this, they used what are called the “quipu” to record quantities of items. (We will describe them in more detail in a moment.) However, they first often needed to do computations whose results would be recorded on quipu. To do these computations, they would sometimes use a counting board constructed with a slab of stone. In the slab were cut rectangular and square compartments so that an octagonal (eight-sided) region was left in the middle. Two opposite corner rectangles were raised. Another two sections were mounted on the original surface of the slab so that there were actually three levels available. In the figure shown, the darkest shaded corner regions represent the highest, third level. The lighter shaded regions surrounding the corners are the second highest levels, while the clear white rectangles are the compartments cut into the stone slab.

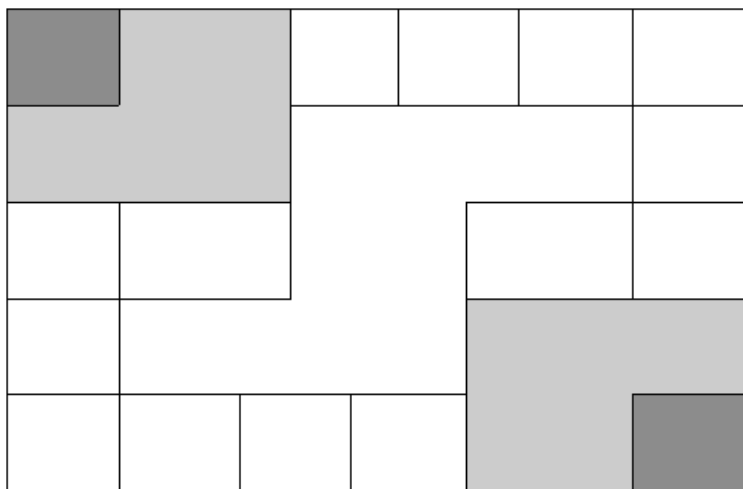


Figure 3.

Pebbles were used to keep accounts and their positions within the various levels and compartments gave totals. For example, a pebble in a smaller (white) compartment represented one unit. Note that there are 12 such squares around the outer edge of the figure. If a pebble was put into one of the two (white) larger, rectangular compartments, its value was doubled. When a pebble was put in the octagonal region in the middle of the slab, its value was tripled. If a pebble was placed on the second (shaded) level, its value was multiplied by six. And finally, if a pebble was found on one of the two highest corner levels, its value was multiplied by twelve. Different objects could be counted at the same time by representing different objects by different colored pebbles.

Example 1

Suppose you have the following counting board with two different kind of pebbles places as illustrated. Let the solid black pebble represent a dog and the striped pebble represent a cat. How many dogs are being represented?

Solution

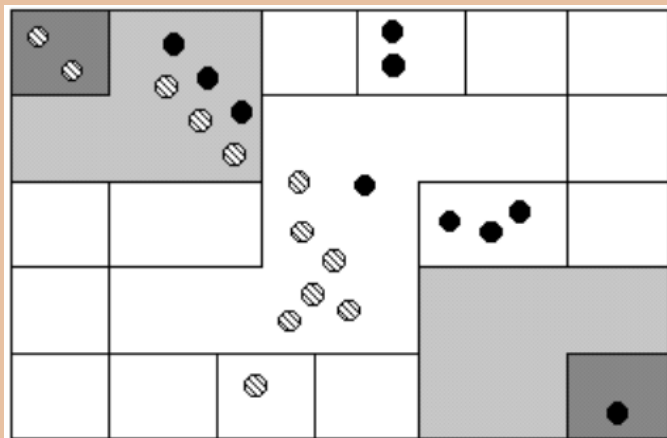


Figure 4.

There are two black pebbles in the outer square regions...these represent 2 dogs. There are three black pebbles in the larger (white) rectangular compartments. These represent 6 dogs. There is one black pebble in the middle region...this represents 3 dogs. There are three

black pebbles on the second level...these represent 18 dogs. Finally, there is one black pebble on the highest corner level...this represents 12 dogs. We then have a total of $2+6+3+18+12 = 41$ dogs.

Try It Now

How many cats are represented on this board?

The Quipu

This kind of board was good for doing quick computations, but it did not provide a good way to keep a permanent recording of quantities or computations. For this purpose, they used the quipu. The quipu is a collection of cords with knots in them. These cords and knots are carefully arranged so that the position and type of cord or knot gives specific information on how to decipher the cord.

A quipu is made up of a main cord which has other cords (branches) tied to it. See pictures to the right.⁷

Locke called the branches H cords. They are attached to the main cord. B cords, in turn, were attached to the H cords. Most of these cords would have knots on them. Rarely are knots found on the main cord, however, and tend to be mainly on the H and B cords. A quipu might also have a “totalizer” cord that summarizes all of the information on the cord group in one place.

Locke points out that there are three types of knots, each representing a different value, depending on the kind of knot used and its position on the cord. The Incas, like us, had a decimal (base-

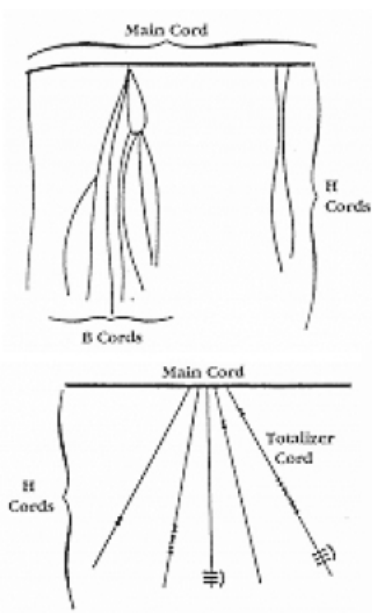


Figure 5.

7. Diana, Lind Mae; *The Peruvian Quipu in Mathematics Teacher*, Issue 60 (Oct., 1967), p. 623–28.

ten) system, so each kind of knot had a specific decimal value. The Single knot, pictured in the middle of figure 6⁸ was used to denote tens, hundreds, thousands, and ten thousands. They would be on the upper levels of the H cords. The figure-eight knot on the end was used to denote the integer “one.” Every other integer from 2 to 9 was represented with a long knot, shown on the left of the figure. (Sometimes long knots were used to represent tens and hundreds.) Note that the long knot has several turns in it...the number of turns indicates which integer is being represented. The units (ones) were placed closest to the bottom of the cord, then tens right above them, then the hundreds, and so on.

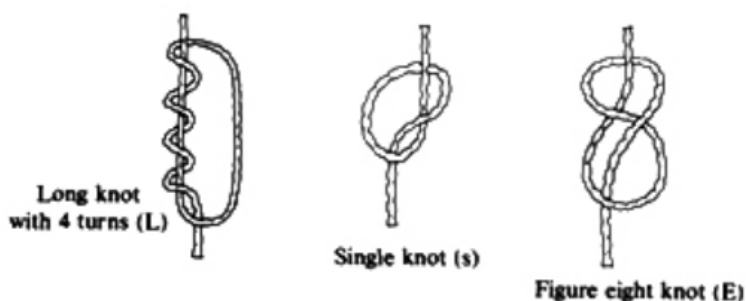


Figure 6

In order to make reading these pictures easier, we will adopt a convention that is consistent. For the long knot with turns in it (representing the numbers 2 through 9), we will use the following notation:

The four horizontal bars represent four turns and the curved arc on the right links the four turns together. This would represent the number 4.

8. <http://wiscinfo.doit.wisc.edu/chaysimire/titulo2/khipus/what.htm>

We will represent the single knot with a large dot (\cdot) and we will represent the figure eight knot with a sideways eight (∞).

Example 2

What number is represented on the cord shown in figure 7?

Solution

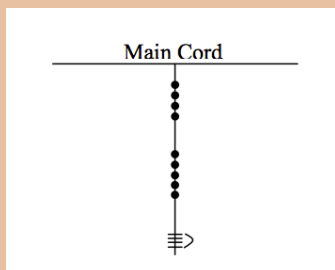


Figure 7.

On the cord, we see a long knot with four turns in it...this represents four in the ones place. Then 5 single knots appear in the tens position immediately above that, which represents 5 tens, or 50. Finally, 4 single knots are tied in the hundreds, representing four 4 hundreds, or 400. Thus, the total shown on this cord is 454.

Try It Now

What numbers are represented on each of the four cords hanging from the main cord?

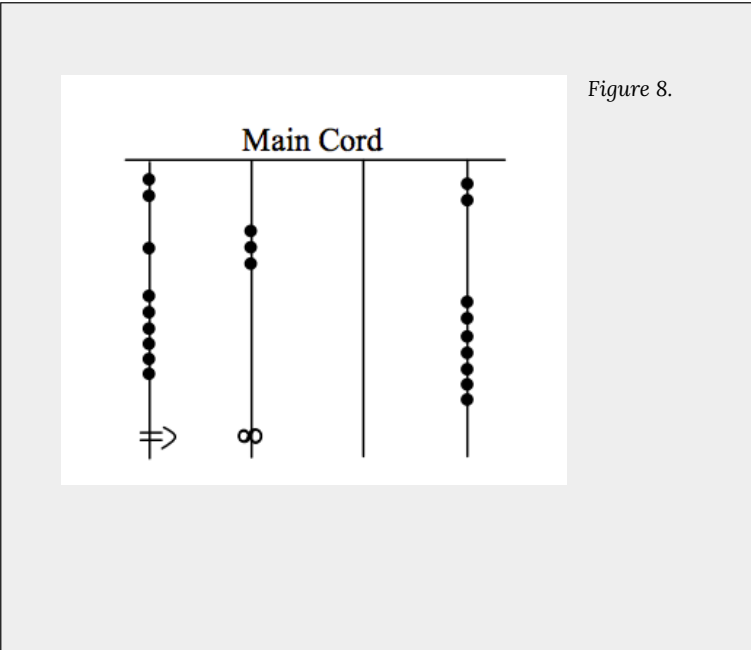


Figure 8.

The colors of the cords had meaning and could distinguish one object from another. One color could represent llamas, while a different color might represent sheep, for example. When all the colors available were exhausted, they would have to be re-used. Because of this, the ability to read the quipu became a complicated task and specially trained individuals did this job. They were called Quipucamayoc, which means keeper of the quipus. They would build, guard, and decipher quipus.

As you can see from this photograph of an actual quipu (figure 9), they could get quite complex.

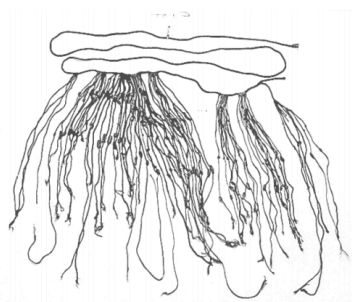


Figure 9.

There were various purposes for the quipu. Some believe that they were used to keep an account of their traditions and history, using knots to record history rather than some other formal system of writing. One

writer has even suggested that the quipu replaced writing as it formed a role in the Incan postal system.⁹ Another proposed use of the quipu is as a translation tool. After the conquest of the Incas by the Spaniards and subsequent “conversion” to Catholicism, an Inca supposedly could use the quipu to confess their sins to a priest. Yet another proposed use of the quipu was to record numbers related to magic and astronomy, although this is not a widely accepted interpretation.

The mysteries of the quipu have not been fully explored yet. Recently, Ascher and Ascher have published a book, *The Code of the Quipu: A Study in Media, Mathematics, and Culture*, which is “an extensive elaboration of the logical-numerical system of the quipu.”¹⁰ For more information on the quipu, you may want to check out “Khipus: a unique Huarochiri legacy.”

We are so used to seeing the symbols 1, 2, 3, 4, etc. that it may be somewhat surprising to see such a creative and innovative way to compute and record numbers. Unfortunately, as we proceed

9. Diana, Lind Mae; *The Peruvian Quipu in Mathematics Teacher*, Issue 60 (Oct., 1967), p. 623–28.

10. <http://www.cs.uidaho.edu/~casey931/seminar/quipu.html>

through our mathematical education in grade and high school, we receive very little information about the wide range of number systems that have existed and which still exist all over the world. That's not to say our own system is not important or efficient. The fact that it has survived for hundreds of years and shows no sign of going away any time soon suggests that we may have finally found a system that works well and may not need further improvement, but only time will tell that whether or not that conjecture is valid or not. We now turn to a brief historical look at how our current system developed over history.

The Hindu—Arabic Number System

The Evolution of a System

Our own number system, composed of the ten symbols {0,1,2,3,4,5,6,7,8,9} is called the *Hindu-Arabic system*. This is a base-ten (decimal) system since place values increase by powers of ten. Furthermore, this system is positional, which means that the position of a symbol has bearing on the value of that symbol within the number. For example, the position of the symbol 3 in the number 435,681 gives it a value much greater than the value of the symbol 8 in that same number. We'll explore base systems more thoroughly later. The development of these ten symbols and their use in a positional system comes to us primarily from India.¹¹

11. http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html

It was not until the fifteenth century that the symbols that we are familiar with today first took form in Europe. However, the history of these numbers and their development goes back hundreds of years. One important source of information on this topic is the writer al-Biruni, whose picture is shown in figure 10.¹² Al-Biruni, who was born in modern day Uzbekistan, had visited India on several occasions and made comments on the Indian



Figure 10. Al-Biruni

number system. When we look at the origins of the numbers that al-Biruni encountered, we have to go back to the third century BCE. to explore their origins. It is then that the Brahmi numerals were being used.

The Brahmi numerals were more complicated than those used in our own modern system. They had separate symbols for the numbers 1 through 9, as well as distinct symbols for 10, 100, 1000,..., also for 20, 30, 40,..., and others for 200, 300, 400, ..., 900. The Brahmi symbols for 1, 2, and 3 are shown below.¹³

12. <http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Al-Biruni.html>

13. http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html

1	2	3
—	=	≡

Brahmi one, two, three

These numerals were used all the way up to the fourth century CE, with variations through time and geographic location. For example, in the first century CE, one particular set of Brahmi numerals took on the following form:¹⁴

1	2	3	4	5	6	7	8	9
—	=	≡	+	h	4	7	5	1

From the fourth century on, you can actually trace several different paths that the Brahmi numerals took to get to different points and incarnations. One of those paths led to our current numeral system, and went through what are called the Gupta numerals. The Gupta numerals were prominent during a time ruled by the Gupta dynasty and were spread throughout that empire as they conquered lands during the fourth through sixth centuries. They have the following form:¹⁵

1	2	3	4	5	6	7	8	9
—	=	≡	4	h	5	7	5	3

How the numbers got to their Gupta form is open to considerable

14. http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html

15. Ibid.

debate. Many possible hypotheses have been offered, most of which boil down to two basic types.¹⁶ The first type of hypothesis states that the numerals came from the initial letters of the names of the numbers. This is not uncommon . . . the Greek numerals developed in this manner. The second type of hypothesis states that they were derived from some earlier number system. However, there are other hypotheses that are offered, one of which is by the researcher Ifrah. His theory is that there were originally nine numerals, each represented by a corresponding number of vertical lines. One possibility is this:¹⁷

1	2	3	4	5	6	7	8	9
I	II	III	II II	III II	III III	III III	III III	III III III

Because these symbols would have taken a lot of time to write, they eventually evolved into cursive symbols that could be written more quickly. If we compare these to the Gupta numerals above, we can try to see how that evolutionary process might have taken place, but our imagination would be just about all we would have to depend upon since we do not know exactly how the process unfolded.

The Gupta numerals eventually evolved into another form of numerals called the Nagari numerals, and these continued to evolve until the eleventh century, at which time they looked like this:¹⁸

16. Ibid.

17. Ibid.

18. Ibid.

1	2	3	4	5	6	7	8	9	0
१	२	३	४	५	६	७	८	९	०

Note that by this time, the symbol for 0 has appeared! The Mayans in the Americas had a symbol for zero long before this, however, as we shall see later in the chapter.

These numerals were adopted by the Arabs, most likely in the eighth century during Islamic incursions into the northern part of India.¹⁹ It is believed that the Arabs were instrumental in spreading them to other parts of the world, including Spain (see below).

Other examples of variations up to the eleventh century include:²⁰



Figure 11.
Devangari,
eighth
century



Figure 12.
West Arab
Gobar, tenth
century



Figure 13.
Spain, 976
BCE

19. Katz, page 230

20. Burton, David M., *History of Mathematics, An Introduction*, p. 254–255

Finally, figure 14²¹ shows various forms of these numerals as they developed and eventually converged to the fifteenth century in Europe.

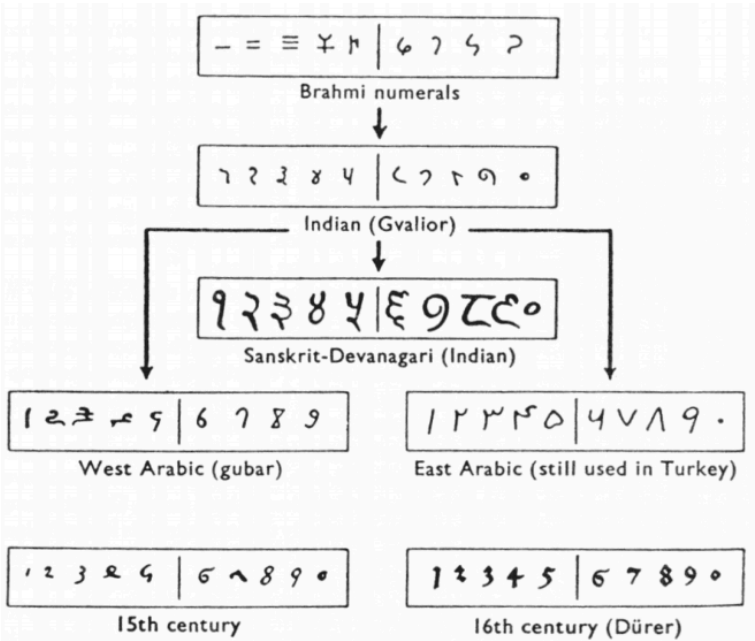


Figure 14.

The Positional System

More important than the form of the number symbols is the development of the place value system. Although it is in slight dispute, the earliest known document in which the Indian system displays a positional system dates back to 346 CE. However, some

21. Katz, page 231.

evidence suggests that they may have actually developed a positional system as far back as the first century CE.

The Indians were not the first to use a positional system. The Babylonians (as we will see in Chapter 3) used a positional system with 60 as their base. However, there is not much evidence that the Babylonian system had much impact on later numeral systems, except with the Greeks. Also, the Chinese had a base-10 system, probably derived from the use of a counting board.²² Some believe that the positional system used in India was derived from the Chinese system.

Wherever it may have originated, it appears that around 600 CE, the Indians abandoned the use of symbols for numbers higher than nine and began to use our familiar system where the position of the symbol determines its overall value.²³ Numerous documents from the seventh century demonstrate the use of this positional system.

Interestingly, the earliest dated inscriptions using the system with a symbol for zero come from Cambodia. In 683, the 605th year of the Saka era is written with three digits and a dot in the middle. The 608th year uses three digits with a modern 0 in the middle.²⁴ The dot as a symbol for zero also appears in a Chinese work (*Chiu-chih li*). The author of this document gives a strikingly clear description of how the Indian system works:

Using the [Indian] numerals, multiplication and division are carried out. Each numeral is written in one stroke. When a number is counted to ten, it is advanced into the higher place. In each vacant place a dot is always put. Thus the numeral is always denoted in each place. Accordingly there

22. Ibid, page 230

23. Ibid, page 231.

24. Ibid, page 232.

can be no error in determining the place. With the numerals, calculations is easy.²⁵

Transmission to Europe

It is not completely known how the system got transmitted to Europe. Traders and travelers of the Mediterranean coast may have carried it there. It is found in a tenth-century Spanish manuscript and may have been introduced to Spain by the Arabs, who invaded the region in 711 CE and were there until 1492.

In many societies, a division formed between those who used numbers and calculation for practical, every day business and those who used them for ritualistic purposes or for state business.²⁶ The former might often use older systems while the latter were inclined to use the newer, more elite written numbers. Competition between the two groups arose and continued for quite some time.

25. Ibid, page 232.

26. McLeish, p. 18

In a fourteenth century manuscript of Boethius' *The Consolations of Philosophy*, there appears a well-known drawing of two mathematicians. One is a merchant and is using an abacus (the "abacist"). The other is a Pythagorean philosopher (the "algorist") using his "sacred" numbers. They are in a competition that is being judged by the goddess of number. By 1500 CE, however, the newer symbols



Figure 15.

and system had won out and has persevered until today. The Seattle Times recently reported that the Hindu-Arabic numeral system has been included in the book *The Greatest Inventions of the Past 2000 Years*.²⁷

One question to answer is *why* the Indians would develop such a positional notation. Unfortunately, an answer to that question is not currently known. Some suggest that the system has its origins with the Chinese counting boards. These boards were portable and it is thought that Chinese travelers who passed through India took their boards with them and ignited an idea in Indian mathematics.²⁸ Others, such as G. G. Joseph propose that it is the Indian fascination with very large numbers that drove them to develop a system whereby these kinds of big numbers could easily

27. http://seattletimes.nwsources.com/news/health-science/html98/invs_20000201.html, Seattle Times, Feb. 1, 2000

28. Ibid, page 232.

be written down. In this theory, the system developed entirely within the Indian mathematical framework without considerable influence from other civilizations.

The Development and Use of Different Number Bases

Introduction and Basics

During the previous discussions, we have been referring to positional base systems. In this section of the chapter, we will explore exactly what a base system is and what it means if a system is “positional.” We will do so by first looking at our own familiar, base-ten system and then deepen our exploration by looking at other possible base systems. In the next part of this section, we will journey back to Mayan civilization and look at their unique base system, which is based on the number 20 rather than the number 10.

A base system is a structure within which we count. The easiest way to describe a base system is to think about our own base-ten system. The base-ten system, which we call the “decimal” system, requires a total of ten different symbols/digits to write any number. They are, of course, $0, 1, 2, \dots, 9$.

The decimal system is also an example of a *positional* base system, which simply means that the position of a digit gives its place value. Not all civilizations had a positional system even though they did have a base with which they worked.

In our base-ten system, a number like 5,783,216 has meaning to us because we are familiar with the system and its places. As we know, there are six ones, since there is a 6 in the ones place. Likewise, there are seven “hundred thousands,” since the 7 resides in that place. Each digit has a value that is explicitly determined by its

position within the number. We make a distinction between digit, which is just a symbol such as 5, and a number, which is made up of one or more digits. We can take this number and assign each of its digits a value. One way to do this is with a table, which follows:

5,000,000	= 5 × 1,000,000	= 5 × 10 ⁶	Five million
+700,000	= 7 × 100,000	= 7 × 10 ⁵	Seven hundred thousand
+80,000	= 8 × 10,000	= 8 × 10 ⁴	Eighty thousand
+3,000	= 3 × 1000	= 3 × 10 ³	Three thousand
+200	= 2 × 100	= 2 × 10 ²	Two hundred
+10	= 1 × 10	= 1 × 10 ¹	Ten
+6	= 6 × 1	= 6 × 10 ⁰	Six
5,783,216	Five million, seven hundred eighty-three thousand, two hundred sixteen		

From the third column in the table we can see that each place is simply a multiple of ten. Of course, this makes sense given that our base is ten. The digits that are multiplying each place simply tell us how many of that place we have. We are restricted to having at most 9 in any one place before we have to “carry” over to the next place. We cannot, for example, have 11 in the hundreds place. Instead, we would carry 1 to the thousands place and retain 1 in the hundreds place. This comes as no surprise to us since we readily see that 11 hundreds is the same as one thousand, one hundred. Carrying is a pretty typical occurrence in a base system.

However, base-ten is not the only option we have. Practically any positive integer greater than or equal to 2 can be used as a base for a number system. Such systems can work just like the decimal system except the number of symbols will be different and each position will depend on the base itself.

Other Bases

For example, let's suppose we adopt a base-five system. The only modern digits we would need for this system are 0,1,2,3 and 4. What are the place values in such a system? To answer that, we start with the ones place, as most base systems do. However, if we were to count in this system, we could only get to four (4) before we had to jump up to the next place. Our base is 5, after all! What is that next place that we would jump to? It would not be tens, since we are no longer in base-ten. We're in a different numerical world. As the base-ten system progresses from 10^0 to 10^1 , so the base-five system moves from 5^0 to $5^1 = 5$. Thus, we move from the ones to the fives.

After the fives, we would move to the 5^2 place, or the twenty fives. Note that in base-ten, we would have gone from the tens to the hundreds, which is, of course, 10^2 .

Let's take an example and build a table. Consider the number 30412 in base five. We will write this as 30412_5 , where the subscript 5 is not part of the number but indicates the base we're using. First off, note that this is NOT the number "thirty thousand, four hundred twelve." We must be careful not to impose the base-ten system on this number. Here's what our table might look like. We will use it to convert this number to our more familiar base-ten system.

Base 5	This column coverts to base-ten	In Base-Ten
3×5^4	$= 3 \times 625$	$= 1875$
$+ 0 \times 5^3$	$= 0 \times 125$	$= 0$
$+ 4 \times 5^2$	$= 4 \times 25$	$= 100$
$+ 1 \times 5^1$	$= 1 \times 5$	$= 5$
$+ 2 \times 5^0$	$= 2 \times 1$	$= 2$
Total		1982

As you can see, the number 30412_5 is equivalent to 1,982 in base-ten. We will say $30412_5 = 1982_{10}$. All of this may seem strange to you, but

that's only because you are so used to the only system that you've ever seen.

Example 3

Convert 6234_7 to a base 10 number.

Solution

We first note that we are given a base-7 number that we are to convert. Thus, our places will start at the ones (7^0), and then move up to the 7s, 49s (7^2), etc. Here's the breakdown:

	Base 7	Convert	Base 10
	$= 6 \times 7^3$	$= 6 \times 343$	$= 2058$
+	$= 2 \times 7^2$	$= 2 \times 49$	$= 98$
+	$= 3 \times 7$	$= 3 \times 7$	$= 21$
+	$= 4 \times 1$	$= 4 \times 1$	$= 4$
		Total	2181

Thus $6234_7 = 2181_{10}$

Try It Now

Convert 41065_7 to a base 10 number.

Converting from Base 10 to Other Bases

Converting from an unfamiliar base to the familiar decimal system is not that difficult once you get the hang of it. It's only a matter of identifying each place and then multiplying each digit by the appropriate power. However, going the other direction can be a little trickier. Suppose you have a base-ten number and you want to convert to base-five. Let's start with some simple examples before we get to a more complicated one.

Example 4

Convert twelve to a base-five number.

Solution

We can probably easily see that we can rewrite this number as follows:

$$12 = (2 \times 5) + (2 \times 1)$$

Hence, we have two fives and 2 ones. Hence, in base-five we would write twelve as 22_5 . Thus, $12_{10} = 22_5$.

Example 5

Convert sixty-nine to a base-five number.

Solution

We can see now that we have more than 25, so we rewrite sixty-nine as follows:

$$69 = (2 \times 25) + (3 \times 5) + (4 \times 1)$$

Here, we have two twenty-fives, 3 fives, and 4 ones. Hence, in base five we have 234. Thus, $69_{10} = 234_5$.

Example 6

Convert the base-seven number 3261_7 to base 10.

Solution

The powers of 7 are:

$$7^0 = 1$$

$$7^1 = 7$$

$$7^2 = 49$$

$$7^3 = 343$$

Etc...

$$3261_7 = (3 \times 343) + (2 \times 49) + (6 \times 7) + (1 \times 1) = 1170_{10}.$$

Thus $3261_7 = 1170_{10}$.

Try It Now

Convert 143 to base 5

Try It Now

Convert the base-three number 21021_3 to base 10.

In general, when converting from base-ten to some other base, it is often helpful to determine the highest power of the base that will divide into the given number at least once.

In the last example, $5^2 = 25$ is the largest power of five that is present in 69, so that was our starting point. If we had moved to $5^3 = 125$, then 125 would not divide into 69 at least once.

Converting from Base 10 to Base b

1. Find the highest power of the base b that will divide into the given number at least once and then divide.
2. Write down the whole number part, then use the remainder from division in the next step.
3. Repeat step two, dividing by the next highest power of the base b , writing down the whole number part (including 0), and using the remainder in the next step.
4. Continue until the remainder is smaller than the

base. This last remainder will be in the “ones” place.

5. Collect all your whole number parts to get your number in base b notation.

Example 7

Convert the base-ten number 348 to base-five.

Solution

The powers of five are:

$$5^0 = 1$$

$$5^1 = 5$$

$$5^2 = 25$$

$$5^3 = 125$$

$$5^4 = 625$$

Etc...

Since 348 is smaller than 625, but bigger than 125, we see that $5^3 = 125$ is the highest power of five present in 348. So we divide 125 into 348 to see how many of them there are:

$$348 \div 125 = 2 \text{ with remainder } 98$$

We write down the whole part, 2, and continue with the remainder. There are 98 left over, so we see how many 25s

(the next smallest power of five) there are in the remainder:

$$98 \div 25 = 3 \text{ with remainder } 23$$

We write down the whole part, 2, and continue with the remainder. There are 23 left over, so we look at the next place, the 5s:

$$23 \div 5 = 4 \text{ with remainder } 3$$

This leaves us with 3, which is less than our base, so this number will be in the “ones” place. We are ready to assemble our base-five number:

$$348 = (2 \times 5^3) + (3 \times 5^2) + (4 \times 5^1) + (3 \times 1)$$

Hence, our base-five number is 2343_5 . We'll say that $348_{10} = 2343_5$.

Example 8

Convert the base-ten number 4,509 to base-seven.

Solution

The powers of 7 are:

$$7^0 = 1$$

$$7^1 = 7$$

$$7^2 = 49$$

$$7^3 = 343$$

$$7^4 = 2401$$

$$7^5 = 16807$$

Etc...

The highest power of 7 that will divide into 4,509 is $7^4 = 2401$. With division, we see that it will go in 1 time with a remainder of 2108. So we have 1 in the 7^4 place.

The next power down is $7^3 = 343$, which goes into 2108 six times with a new remainder of 50. So we have 6 in the 7^3 place.

The next power down is $7^2 = 49$, which goes into 50 once with a new remainder of 1. So there is a 1 in the 7^2 place.

The next power down is 7^1 but there was only a remainder of 1, so that means there is a 0 in the 7s place and we still have 1 as a remainder.

That, of course, means that we have 1 in the ones place.

Putting all of this together means that $4,509_{10} = 16101_7$.

$$4,509 \div 7^4 = 1 \text{ R } 2108 \quad 2108 \div 7^3 = 6 \text{ R } 50$$

$$50 \div 7^2 = 1 \text{ R } 1$$

$$1 \div 7^1 = 1$$

$$4,509_{10} = 16101_7.$$

Try It Now

Convert 657_{10} to a base 4 number.

Try It Now

Convert 8377_{10} to a base 8 number.

Another Method For Converting From Base 10 to Other Bases

As you read the solution to this last example and attempted the “Try it Now” problems, you may have had to repeatedly stop and think about what was going on. The fact that you are probably struggling to follow the explanation and reproduce the process yourself is mostly due to the fact that the non-decimal systems are so unfamiliar to you. In fact, the only system that you are probably comfortable with is the decimal system.

As budding mathematicians, you should always be asking questions like “How could I simplify this process?” In general, that is

one of the main things that mathematicians do: they look for ways to take complicated situations and make them easier or more familiar. In this section we will attempt to do that.

To do so, we will start by looking at our own decimal system. What we do may seem obvious and maybe even intuitive but that's the point. We want to find a process that we readily recognize works and makes sense to us in a familiar system and then use it to extend our results to a different, unfamiliar system.

Let's start with the decimal number, 4863_{10} . We will convert this number to base 10. Yeah, I know it's already in base 10, but if you carefully follow what we're doing, you'll see it makes things work out very nicely with other bases later on. We first note that the highest power of 10 that will divide into 4863 at least once is $10^3 = 1000$. *In general, this is the first step in our new process; we find the highest power that a given base that will divide at least once into our given number.*

We now divide 1000 into 4863:

$$4863 \div 1000 = 4.863$$

This says that there are four thousands in 4863 (obviously). However, it also says that there are 0.863 thousands in 4863. This fractional part is our remainder and will be converted to lower powers of our base (10). If we take that decimal and multiply by 10 (since that's the base we're in) we get the following:

$$0.863 \times 10 = 8.63$$

Why multiply by 10 at this point? We need to recognize here that 0.863 thousands is the same as 8.63 hundreds. Think about that until it sinks in.

$$(0.863)(1000) = 863$$

$$(8.63)(100) = 863$$

These two statements are equivalent. So, what we are really doing here by multiplying by 10 is rephrasing or converting from one place (thousands) to the next place down (hundreds).

$$0.863 \times 10 \Rightarrow 8.63$$

$$(\text{Parts of Thousands}) \times 10 \Rightarrow \text{Hundreds}$$

What we have now is 8 hundreds and a remainder of 0.63

hundreds, which is the same as 6.3 tens. We can do this again with the 0.63 that remains after this first step.

$$0.63 \times 10 \Rightarrow 6.3$$

$$\text{Hundreds} \times 10 \Rightarrow \text{Tens}$$

So we have six tens and 0.3 tens, which is the same as 3 ones, our last place value.

Now here's the punch line. Let's put all of the together in one place:

$$\begin{array}{rcl}
 4863 \div 1000 = & \textcircled{4} & .863 \\
 \downarrow & & \downarrow \\
 0.863 \times 10 = & \textcircled{8} & .63 \\
 \downarrow & & \downarrow \\
 0.63 \times 10 = & \textcircled{6} & .3 \\
 \downarrow & & \downarrow \\
 0.3 \times 10 = & \textcircled{3} & .0
 \end{array}$$

Converting from Base 10 to Base b : Another methodNote that in each step, the remainder is carried down to the next step and multiplied by 10, the base. Also, at each step, the whole number part, which is circled, gives the digit that belongs in that particular place. What is amazing is that this works for any base! So, to convert from a base 10 number to some other base, b , we have the following steps we can follow:

Converting from Base 10 to Base b :

Another method

1. Find the highest power of the base b that will divide into the given number at least once and then divide.
2. Keep the whole number part, and multiply the fractional part by the base b .
3. Repeat step two, keeping the whole number part (including 0), carrying the fractional part to the next step until only a whole number result is obtained.
4. Collect all your whole number parts to get your number in base b notation.

We will illustrate this procedure with some examples.

Example 9

Convert the base 10 number, 348_{10} , to base 5.

Solution

This is actually a conversion that we have done in a previous example. The powers of five are:

$$5^0 = 1$$

$$5^1 = 5$$

$$5^2 = 25$$

$$5^3 = 125$$

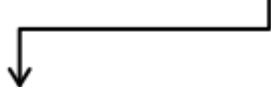
$$5^4 = 625$$

Etc...

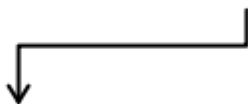
The highest power of five that will go into 348 at least once is 5^3 .

We divide by 125 and then proceed.

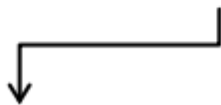
$$348 \div 5^3 = \textcircled{2}.784$$



$$0.784 \times 5 = \textcircled{3}.92$$



$$0.92 \times 5 = \textcircled{4}0.6$$



$$0.6 \times 5 = \textcircled{3}.0$$

By keeping all the whole number parts, from top bottom, gives 2343 as our base 5 number. Thus, $2343_5 = 348_{10}$.

We can compare our result with what we saw earlier, or simply check with our calculator, and find that these two numbers really are equivalent to each other.

Example 10

Convert the base 10 number, 3007_{10} , to base 5.

Solution

The highest power of 5 that divides at least once into 3007 is $5^4 = 625$. Thus, we have:

$$3007 \div 625 = \textcircled{4}.8112$$

$$0.8112 \times 5 = \textcircled{4}.056$$

$$0.056 \times 5 = \textcircled{0}.28$$

$$0.28 \times 5 = \textcircled{1}0.4$$

$$0.4 \times 5 = \textcircled{2}0.0$$

This gives us that $3007_{10} = 44012_5$. Notice that in the third line that multiplying by 5 gave us 0 for our whole number part. We don't discard that! The zero tells us that a zero in that place. That is, there are no 5^2 s in this number.

This last example shows the importance of using a calculator in certain situations and taking care to avoid clearing the calculator's memory or display until you get to the very end of the process.

Example 11

Convert the base 10 number, 63201_{10} , to base 7.

Solution

The powers of 7 are:

$$7^0 = 1$$

$$7^1 = 7$$

$$7^2 = 49$$

$$7^3 = 343$$

$$7^4 = 2401$$

$$7^5 = 16807$$

etc...

The highest power of 7 that will divide at least once into 63201 is 7^5 . When we do the initial division on a calculator, we get the following:

$$63201 \div 7^5 = 3.760397453$$

The decimal part actually fills up the calculators display and we don't know if it terminates at some point or perhaps even repeats down the road. So if we clear our calculator at this point, we will introduce error that is likely to keep this process from ever ending. To avoid this problem, we leave the result in the calculator and simply subtract 3 from this to get the fractional part all by itself. **Do not round off!**

Subtraction and then multiplication by seven gives:

$$63201 \div 7^5 = f\textcircled{3}.760397453$$

$$0.760397453 \times 7 = \dots\textcircled{5}.322782174$$

$$0.322782174 \times 7 =, \textcircled{2}.259475219$$

$$0.259475219 \times 7 =? \textcircled{1}.816326531$$

$$0.816326531 \times 7 = \textcircled{5}....714285714$$

$$0.714285714 \times 7 = \dots\textcircled{5}.000000000$$

Yes, believe it or not, that last product is exactly 5, *as long as you don't clear anything out on your calculator*. This gives us our final result: $63201_{10} = 352155_7$.

If we round, even to two decimal places in each step, clearing our calculator out at each step along the way, we will get a series of numbers that do not terminate, but begin repeating themselves endlessly. (Try it!) We end up with something that doesn't make any sense, at least not in this context. So be careful to use your calculator cautiously on these conversion problems.

Also, remember that if your first division is by 7^5 , then you expect to have 6 digits in the final answer, corresponding to the places for 7^5 , 7^4 , and so on down to 7^0 . If you find yourself with more than 6 digits due to rounding errors, you know something went wrong.

Try It Now

Convert the base 10 number, 9352_{10} , to base 5.

Try It Now

Convert the base 10 number, 1500, to base 3.

Be careful not to clear your calculator on this one. Also, if you're not careful in each step, you may not get all of the digits you're looking for, so move slowly and with caution.

The Mayan Numeral System

Background

As you might imagine, the development of a base system is an important step in making the counting process more efficient. Our own base-ten system probably arose from the fact that we have 10 fingers (including thumbs) on two hands. This is a natural development. However, other civilizations have had a variety of bases other than ten. For example, the Natives of Queensland used a base-two system, counting as follows: “one, two, two and one, two two’s, much.” Some Modern South American Tribes have a base-five system counting in this way: “one, two, three, four, hand, hand and one, hand and two,” and so on. The Babylonians used a base-sixty (sexagesimal) system. In this chapter, we wrap up with a specific example of a civilization that actually used a base system other than 10.



1. [Chichen Itza](#)

2. [Uxmal](#)

3. [Tulum](#)

4. [Palenque](#)

5. [Bonampak, Yaxchilan](#)

6. [Tikal](#)

7. [Altun Ha](#)

8. [Copán](#)

The Mayan civilization is generally dated from 1500 BCE to 1700 CE. The Yucatan Peninsula (see figure 16²⁹) in Mexico was the scene for the development of one of the most advanced civilizations of the ancient world. The Mayans had a sophisticated ritual system that was overseen by a priestly class. This class of priests developed a

29. http://www.gorp.com/gorp/location/latamer/map_maya.htm

philosophy with time as divine and eternal.³⁰ The calendar, and calculations related to it, were thus very important to the ritual life of the priestly class, and hence the Mayan people. In fact, much of what we know about this culture comes from their calendar records and astronomy data. Another important source of information on the Mayans is the writings of Father Diego de Landa, who went to Mexico as a missionary in 1549.



There were two numeral systems developed by the Mayans—one for the common people and one for the priests. Not only did these two systems use different symbols, they also used different base systems. For the priests, the number system was governed by ritual. The days of the year were thought to be gods, so the formal symbols for the days were decorated heads,³¹ like the sample to the left³² Since the basic calendar was based on 360 days, the priestly numeral system used a mixed base system employing multiples of 20 and 360. This makes for a confusing system, the details of which we will skip.

30. Bidwell, James; Mayan Arithmetic in *Mathematics Teacher*, Issue 74 (Nov., 1967), p. 762–68.





















31. <http://www.ukans.edu/~lctls/Mayan/numbers.html>

32. <http://www.ukans.edu/~lctls/Mayan/numbers.html>

Powers	Base-Ten Value	Place Name
20^7	12,800,000,000	Hablat
20^6	64,000,000	Alau
20^5	3,200,000	Kinchil
20^4	160,000	Cabal
20^3	8,000	Pic
20^2	400	Bak
20^1	20	Kal
20^0	1	Hun

The Mayan Number System

Instead, we will focus on the numeration system of the “common” people, which used a more consistent base system. As we stated earlier, the Mayans used a base-20 system, called the “vigesimal” system. Like our system, it is positional, meaning that the position of a numeric symbol indicates its place value. In the following table you can see the place value in its vertical format.³³

Number	Vertical Form	Number	Vertical Form
0		10	
1		11	
2		12	
3		13	
4		14	
5		15	
6		16	
7		17	
8		18	
9		19	

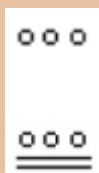
In order to write numbers down, there were only three symbols needed in this system. A horizontal bar represented the quantity 5, a dot represented the quantity 1, and a special symbol (thought to be a shell) represented zero. The Mayan system may have been the first to make use of zero as a placeholder/number. The first 20 numbers are shown in the table to the right.³⁴

Unlike our system, where the ones place starts on the right and then moves to the left, the Mayan systems places the ones on the **bottom** of a vertical orientation and moves up as the place value increases.

When numbers are written in vertical form, there should never be more than four dots in a single place. When writing Mayan numbers, every group of five dots becomes one bar. Also, there should never be more than three bars in a single place...four bars would be converted to one dot in the next place up. It's the same as 10 getting converted to a 1 in the next place up when we carry during addition.

Example 12

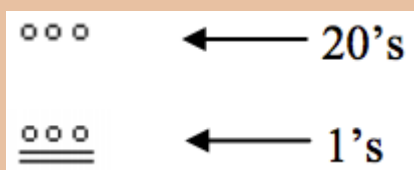
What is the value of this number, which is shown in vertical form?



34. http://www.vpds.wsu.edu/fair_95/gym/UM001.html

Solution

Starting from the bottom, we have the ones place. There are two bars and three dots in this place. Since each bar is worth 5, we have 13 ones when we count the three dots in the ones place. Looking to the place value above it (the twenties places), we see there are three dots so we have three twenties.



Hence we can write this number in base-ten as:

$$(3 \times 20^1) + (13 \times 20^0) = (3 \times 20^1) + (13 \times 1) = 60 + 13 = 73$$

Example 13

What is the value of the following Mayan number?



Solution

This number has 11 in the ones place, zero in the 20s place, and 18 in the $20^2 = 400$ s place. Hence, the value of this number in base-ten is:

$$18 \times 400 + 0 \times 20 + 11 \times 1 = 7211.$$

Try It Now

Convert the Mayan number below to base 10.



Example 14

Convert the base 10 number 3575_{10} to Mayan numerals.

This problem is done in two stages. First we need to convert to a base 20 number. We will do so using the method provided in the last section of the text. The second step is to convert that number to Mayan symbols.

The highest power of 20 that will divide into 3575 is $20^2 = 400$, so we start by dividing that and then proceed from there:

$$3575 \div 400 = 8.9375$$

$$0.9375 \times 20 = 18.75$$

$$0.75 \times 20 = 15.0$$

This means that $3575_{10} = 8,18,15_{20}$

The second step is to convert this to Mayan notation. This number indicates that we have 15 in the ones position. That's three bars at the bottom of the number. We also have 18 in the 20s place, so that's three bars and three dots in the second position. Finally, we have 8 in the 400s place, so that's one bar and three dots on the top. We get the following:



Note that in the previous example a new notation was used when we wrote $8,18,15_{20}$. The commas between the three numbers 8, 18, and

15 are now separating place values for us so that we can keep them separate from each other. This use of the comma is slightly different than how they're used in the decimal system. When we write a number in base 10, such as 7,567,323, the commas are used primarily as an aide to read the number easily but they do not separate single place values from each other. We will need this notation whenever the base we use is larger than 10.

Writing numbers with bases bigger than 10

When the base of a number is larger than 10, separate each “digit” with a comma to make the separation of digits clear.

For example, in base 20, to write the number corresponding to $17 \times 20^2 + 6 \times 20^1 + 13 \times 20^0$, we'd write 17,6,13₂₀.

Try It Now

Convert the base 10 number 10553₁₀ to Mayan numerals.

Try It Now

Convert the base 10 number 5617_{10} to Mayan numerals.

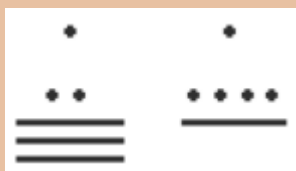
Adding Mayan Numbers

When adding Mayan numbers together, we'll adopt a scheme that the Mayans probably did not use but which will make life a little easier for us.

Example 15

Add, in Mayan, the numbers 37 and 29.³⁵

35. <http://forum.swarthmore.edu/k12/mayan.math/mayan2.html>



First draw a box around each of the vertical places. This will help keep the place values from being mixed up.

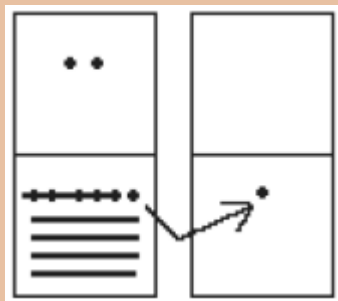


Next, put all of the symbols from both numbers into a single set of places (boxes), and to the right of this new number draw a set of empty boxes where you will place the final sum:



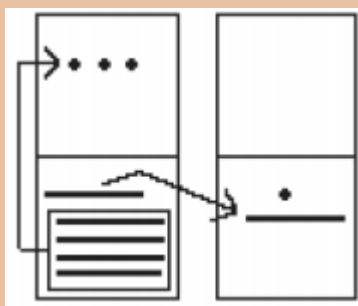
You are now ready to start carrying. Begin with the place that has the lowest value, just as you do with Arabic numbers. Start at the bottom place, where each dot is

worth 1. There are six dots, but a maximum of four are allowed in any one place; once you get to five dots, you must convert to a bar. Since five dots make one bar, we draw a bar through five of the dots, leaving us with one dot which is under the four-dot limit. Put this dot into the bottom place of the empty set of boxes you just drew:

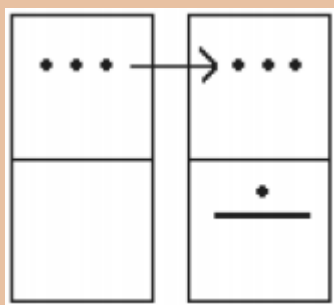


Now look at the bars in the bottom place. There are five, and the maximum number the place can hold is three. *Four bars are equal to one dot in the next highest place.*

Whenever we have four bars in a single place we will automatically convert that to a *dot* in the next place up. We draw a circle around four of the bars and an arrow up to the dots' section of the higher place. At the end of that arrow, draw a new dot. That dot represents 20 just the same as the other dots in that place. Not counting the circled bars in the bottom place, there is one bar left. One bar is under the three-bar limit; put it under the dot in the set of empty places to the right.



Now there are only three dots in the next highest place, so draw them in the corresponding empty box.



We can see here that we have 3 twenties (60), and 6 ones, for a total of 66. We check and note that $37 + 29 = 66$, so we have done this addition correctly. Is it easier to just do it in base-ten? Probably, but that's only because it's more familiar to you. Your task here is to try to learn a new base system and how addition can be done in slightly different ways than what you have seen in the past. Note, however, that the concept of carrying is still used, just as it is in our own addition algorithm.

Try It Now

Try adding 174 and 78 in Mayan by first converting to Mayan numbers and then working entirely within that system. Do not add in base-ten (decimal) until the very end when you *check* your work.

Conclusion

In this reading, we have briefly sketched the development of numbers and our counting system, with the emphasis on the “brief” part. There are numerous sources of information and research that fill many volumes of books on this topic. Unfortunately, we cannot begin to come close to covering all of the information that is out there.

We have only scratched the surface of the wealth of research and information that exists on the development of numbers and counting throughout human history. What is important to note is that the system that we use every day is a product of thousands of years of progress and development. It represents contributions by many civilizations and cultures. It does not come down to us from the sky, a gift from the gods. It is not the creation of a textbook publisher. It is indeed as human as we are, as is the rest of mathematics. Behind every symbol, formula and rule there is a human face to be found, or at least sought.

Furthermore, we hope that you now have a basic appreciation for just how interesting and diverse number systems can get. Also,

we're pretty sure that you have also begun to recognize that we take our own number system for granted so much that when we try to adapt to other systems or bases, we find ourselves truly having to concentrate and think about what is going on.

7. Supplemental Videos



This YouTube playlist contains several videos that supplement the reading on Historical Counting.

You are not required to watch all of these videos, but I recommend watching the videos for any concepts you may be struggling with.

PART V

MODULE 4: PROBABILITY

8. Calculating the Odds of an Event

Here you'll calculate odds by using outcomes or probability. Have you ever thought about the likelihood of an event happening? Take a look at this dilemma:



Telly and Carey were already hard at work when Ms. Kelley came into the bike shop on Thursday morning. It was three days before the big race and there was still a lot of work to be done.

"I can't believe it!" Ms. Kelley exclaimed as she came into the shop.

"What?" both girl asked alarmed.

"There is a 4 to 5 chance that it is going to rain on Saturday. I just heard the weather report," Ms. Kelley said sighing.

"Well, there is still a chance that it won't," Telly said trying to cheer her up.

When we think about chances and odds, we can calculate the likelihood that an event will or won't occur. In this case, there are odds that it will rain and odds that it won't. We can also express those odds as a fraction or a percentage. Learn about odds in this reading, and you can work on the odds of the rainstorm at the end.

Guidance

You've seen that the probability of an event is defined as a ratio that compares the favorable outcomes to the total outcomes. We can write this ratio in fraction form.

$$P(\text{event}) = \frac{\text{favorable outcomes}}{\text{total outcomes}}$$

Sometimes people express the likelihood of events in terms of **odds** rather than probabilities. The odds of an event occurring are equal to the ratio of favorable outcomes to **unfavorable outcomes**.



Think about the odds for the arrow of the spinner above landing on red:

- favorable outcomes = 1(red)
- unfavorable outcomes = 2(blue, yellow)
- total outcomes = 3

So the *probability* of spinning red is:

$$P(\text{red}) = \frac{\text{favorable outcomes}}{\text{total outcomes}} = \frac{1}{3}$$

While the *odds* in favor of red are:

$$\text{Odds(in favor of red)} = \frac{\text{favorable outcomes}}{\text{unfavorable outcomes}} = \frac{1}{2}$$

Odds against an event occurring are defined as:

$$\text{Odds(against red)} = \frac{\text{unfavorable outcomes}}{\text{favorable outcomes}} = \frac{2}{1}$$

You can solve any probability problem in terms of odds rather than probabilities. Notice that the ratio represents what is being compared. Be sure that your numbers match the comparison.

We can use odds to calculate how likely an event is to happen. We can compare the odds in favor of an event with the probability that the event will actually occur. Let's look at an example.

Take a look at this situation.

You've seen that the odds in favor of an event (*E*) occurring are shown in this ratio.

$$\text{Odds(in favor of)} E = \frac{\text{favorable outcomes}}{\text{unfavorable outcomes}} = \frac{1}{2}$$

And the odds against the same event occurring are:

$$\text{Odds(against)} E = \frac{\text{unfavorable outcomes}}{\text{favorable outcomes}} = \frac{2}{1}$$

You can use these two facts to compute the ratio of things happening and not happening.



For example, suppose the weather forecast states:

Odds in favor of rain: 7 to 3

These odds tell you not only the odds of rain, but also the odds of not raining.

If the odds in favor of rain are 7 to 3, then the odds against rain are:

Odds against rain: 3 to 7

Another way of saying that is:

Odds that it will NOT rain: 3 to 7

You can use this idea in many different situations. If you know the odds that something will happen, then you also know the odds that it will not happen.

Use this spinner to calculate odds.



Example A

Odds in favor of spinning a blue.

Solution:

$$\frac{1}{2}$$

Example B

Odds in favor of spinning a red or blue.

Solution:

$$\frac{2}{1}$$

Example C

Odds against spinning a red or blue.

Solution:

$$\frac{1}{2}$$

Intro Problem Revisited

Now let's go back to the dilemma from the beginning of the reading.

Answer all three questions.

What are the chances that it won't rain? We know that the odds of it raining is 4 to 5. Therefore it is a 1 out of 5 chance that it won't rain. Not very good odds.

What are the odds that it will as a percentage? 4 to 5 can be written as a percentage: 80% chance of rain.

What are the odds that it won't as a percentage? 1 to 5 can be written as a percentage: 20% chance that it won't rain.

Guided Practice

Here is one for you to try on your own.

What are the odds in favor of a number cube landing on 4?

Step 1

Find the favorable and unfavorable outcomes.

- favorable outcomes = 1(4)
- unfavorable outcomes = 5(1,2,3,5,6)

Step 2

Write the ratio of favorable to unfavorable outcomes.

$$\text{Odds}(4) = \frac{\text{favorable outcomes}}{\text{unfavorable outcomes}} = \frac{1}{5}$$

The odds in favor of rolling a 4 are 1 to 5.

Vocabulary

Disjoint events: events that don't have any outcomes in common.

Complementary events: probability that has a sum of 100%. Either/Or events are complementary events.

Watch This: Video Review

Probability to Odds

Odds in favor of $A = \frac{P(A)}{P(\text{not } A)}$ Odds against $A = \frac{P(\text{not } A)}{P(A)}$

Find the odds in favor of rolling a 1.

$$\text{Odds} = \frac{P(1)}{P(\text{not } 1)} = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5} = 1 : 5 = 1 \text{ to } 5$$

Find the odds against rolling a 5 or 6.

$$\text{Odds} = \frac{P(\text{not } 5 \text{ or not } 6)}{P(5 \text{ or } 6)} = \frac{\frac{4}{6}}{\frac{2}{6}} = \frac{4}{2} = \frac{2}{1} = 2 : 1 = 2 \text{ to } 1$$

Six Sided Die:
1, 2, 3, 4, 5, 6

A YouTube element has been excluded from this version of the text. You can view it online here:
<https://library.achievingthedream.org/hostosintrocollegemath/?p=29>

Practice Questions

Solve the problems.

1. For rolling a number cube, what are the odds in favor of rolling

a 2?

2. For rolling a number cube, what are the odds against rolling a 2?
3. For rolling a number cube, what are the odds in favor of rolling a number greater than 3?
4. For rolling a number cube, what are the odds in favor rolling a number less than 5?
5. For rolling a number cube, what are the odds against rolling a number less than 5?
6. For rolling a number cube, what are the odds in favor of rolling an even number?
7. For rolling a number cube, what are the odds against rolling an even number?

For a spinner numbered 1 –10, answer the following questions.

1. For spinning the spinner, what are the odds in favor of the arrow landing on 10?
2. For spinning the spinner, what are the odds in favor of the arrow landing on a 2 or 3?
3. For spinning the spinner, what are the odds in favor of the arrow landing on 7, 8 or 9?
4. For spinning the spinner, what are the odds in favor of NOT landing on an even number?
5. For spinning the spinner, what are the odds of the arrow NOT landing on 10?
6. For spinning the spinner, what are the odds in favor of the arrow landing on a number greater than 2?
7. For spinning the spinner, what are the odds in favor of the arrow NOT landing on a number greater than 2?
8. For spinning the spinner, what are the odds of the arrow not landing on a number greater than 3?

9. Probability Reading II

We can use permutations and combinations to help us answer more complex probability questions

Example 1

A 4 digit PIN is selected. What is the probability that there are no repeated digits?

There are 10 possible values for each digit of the PIN (namely: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), so there are $10 \times 10 \times 10 \times 10 = 10^4 = 10000$ total possible PINs.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute $10 \times 9 \times 8 \times 7$, or notice that this is the same as the permutation ${}_{10}P_4 = 5040$.

The probability of no repeated digits is the number of 4 digit PINs with no repeated digits divided by the total number of 4 digit PINs. This probability is

$$\frac{{}_{10}P_4}{10^4} = \frac{5040}{10000} = 0.504$$

Example 2

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. In this lottery, the order the numbers are drawn in doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

In order to compute the probability, we need to count the total number of ways six numbers can be drawn, and the number of ways the six numbers on the player's ticket could match the six numbers drawn from the machine. Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$. Of these possible outcomes, only one would match all six numbers on the player's ticket, so the probability of winning the grand prize is:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815$$

Example 3

In the state lottery from the previous example, if five of the six numbers drawn match the numbers that a player

has chosen, the player wins a second prize of \$1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

As above, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$. In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers; in other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers. The number of ways to choose 5 out of the 6 winning numbers is given by ${}_6C_5 = 6$ and the number of ways to choose 1 out of the 42 losing numbers is given by ${}_{42}C_1 = 42$. Thus the number of favorable outcomes is then given by the Basic Counting Rule: ${}_6C_5 \times {}_{42}C_1 = 6 \times 42 = 252$. So the probability of winning the second prize is

$$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205$$

Try it Now

A multiple-choice question on an economics quiz contains 10 questions with five possible answers each. Compute the probability of randomly guessing the answers and getting 9 questions correct.

Example 4

Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

In many card games (such as poker) the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses); in the problems that follow, we will assume that this is the case unless otherwise stated. Thus we use combinations to compute the possible number of 5-card hands, ${}_{52}C_5$. This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one Ace and four other cards (none of them Aces) from the deck. Since there are four Aces and we want exactly one of them, there will be ${}_4C_1$ ways to select one Ace; since there are 48 non-Aces and we want 4 of them, there will be ${}_{48}C_4$ ways to select the four non-Aces. Now we use the Basic Counting Rule to calculate that there will be ${}_4C_1 \times {}_{48}C_4$ ways to choose one ace and four non-Aces.

Putting this all together, we have

$$P(\text{one Ace}) = \frac{({}_4C_1)({}_{48}C_4)}{{}_{52}C_5} = \frac{778320}{2598960} \approx 0.299$$

Example 5

Compute the probability of randomly drawing five cards from a deck and getting exactly two Aces.

The solution is similar to the previous example, except now we are choosing 2 Aces out of 4 and 3 non-Aces out of 48; the denominator remains the same:

$$P(\text{two Ace}) = \frac{{}_4C_2({}_{48}C_3)}{{}_{52}C_5} = \frac{103776}{2598960} \approx 0.0399$$

It is useful to note that these card problems are remarkably similar to the lottery problems discussed earlier.

Try it Now

Compute the probability of randomly drawing five cards from a deck of cards and getting three Aces and two Kings.

Birthday Problem

Let's take a pause to consider a famous problem in probability theory:

Suppose you have a room full of 30 people. What is the probability that there is at least one shared birthday?

Take a guess at the answer to the above problem. Was your guess fairly low, like around 10%? That seems to be the intuitive answer (30/365, perhaps?). Let's see if we should listen to our intuition. Let's start with a simpler problem, however.

Example 6

Suppose three people are in a room. What is the probability that there is at least one shared birthday among these three people?

There are a lot of ways there could be at least one shared birthday. Fortunately there is an easier way. We ask ourselves "What is the alternative to having at least one shared birthday?" In this case, the alternative is that there are **no** shared birthdays. In other words, the alternative to "at least one" is having **none**. In other words, since this is a complementary event,

$$P(\text{at least one}) = 1 - P(\text{none})$$

We will start, then, by computing the probability that there is no shared birthday. Let's imagine that you are one of these three people. Your birthday can be anything without conflict, so there are 365 choices out of 365 for your birthday. What is the probability that the second person does not share your birthday? There are 365 days in

the year (let's ignore leap years) and removing your birthday from contention, there are 364 choices that will guarantee that you do not share a birthday with this person, so the probability that the second person does not share your birthday is $364/365$. Now we move to the third person. What is the probability that this third person does not have the same birthday as either you or the second person? There are 363 days that will not duplicate your birthday or the second person's, so the probability that the third person does not share a birthday with the first two is $363/365$.

We want the second person not to share a birthday with you *and* the third person not to share a birthday with the first two people, so we use the multiplication rule:

$$P(\text{no shared birthday}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \approx 0.9918$$

and then subtract from 1 to get

$$P(\text{shared birthday}) = 1 - P(\text{no shared birthday}) = 1 - 0.9918 = 0.0082.$$

This is a pretty small number, so maybe it makes sense that the answer to our original problem will be small. Let's make our group a bit bigger.

Example 7

Suppose five people are in a room. What is the

probability that there is at least one shared birthday among these five people?

Continuing the pattern of the previous example, the answer should be

$$P(\text{shared birthday}) = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \approx 0.0271$$

Note that we could rewrite this more compactly as

$$P(\text{shared birthday}) = 1 - \frac{365P_5}{365^5} \approx 0.0271$$

which makes it a bit easier to type into a calculator or computer, and which suggests a nice formula as we continue to expand the population of our group.

Example 8

Suppose 30 people are in a room. What is the probability that there is at least one shared birthday among these 30 people?

Here we can calculate

$$P(\text{shared birthday}) = 1 - \frac{365P_{30}}{365^{30}} \approx 0.706$$

which gives us the surprising result that when you are in a room with 30 people there is a 70% chance that there will be at least one shared birthday!

If you like to bet, and if you can convince 30 people to reveal their

birthdays, you might be able to win some money by betting a friend that there will be at least two people with the same birthday in the room anytime you are in a room of 30 or more people. (Of course, you would need to make sure your friend hasn't studied probability!) You wouldn't be guaranteed to win, but you should win more than half the time.

This is one of many results in probability theory that is counterintuitive; that is, it goes against our gut instincts. If you still don't believe the math, you can carry out a simulation. Just so you won't have to go around rounding up groups of 30 people, someone has kindly developed a Java applet so that you can conduct a computer simulation. Go to this web page, and once the applet has loaded, select 30 birthdays and then keep clicking Start and Reset. If you keep track of the number of times that there is a repeated birthday, you should get a repeated birthday about 7 out of every 10 times you run the simulation.

Try it Now

Suppose 10 people are in a room. What is the probability that there is at least one shared birthday among these 10 people?

Expected Value



Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it's one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

Example 9

In the casino game roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets \$1 on a single number. If that number is spun on the wheel, then they receive \$36 (their original \$1 + \$35). Otherwise, they lose their \$1. On average, how much money should a player expect to win or lose if they play this game repeatedly?

Suppose you bet \$1 on each of the 38 spaces on the wheel, for a total of \$38 bet. When the winning number is spun, you are paid \$36 on that number. While you won on

that one number, overall you've lost \$2. On a per-space basis, you have “won”— $\$2/\$38 \approx -\$0.053$. In other words, on average you lose 5.3 cents per space you bet on.

We call this average gain or loss the expected value of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact, about 37 out of every 38) lose \$1 and a very few people (about 1 person out of every 38) gain \$35 (the \$36 they win minus the \$1 they spent to play the game).

There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is $\frac{1}{38}$. The complement, the probability of losing, is

$$\frac{37}{38}.$$

Summarizing these along with the values, we get this table:

Outcome	Probability of outcome
\$35	$\frac{1}{38}$
-\$1	$\frac{37}{38}$

Notice that if we multiply each outcome by its corresponding probability we get $\$35 \times \frac{1}{38} = 0.9211$ and $-\$1 \times \frac{37}{38} = -0.9737$, and if we add these numbers we get $0.9211 + (-0.9737) \approx -0.053$, which is the expected value we computed above.

Expected Value

Expected Value is the average gain or loss of an event if the procedure is repeated many times.

We can compute the expected value by multiplying each outcome by the probability of that outcome, then adding up the products.

Try it Now

You purchase a raffle ticket to help out a charity. The raffle ticket costs \$5. The charity is selling 2000 tickets. One of them will be drawn and the person holding the ticket will be given a prize worth \$4000. Compute the expected value for this raffle.

Example 10

In a certain state's lottery, 48 balls numbered 1 through

48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. If they match 5 numbers, then win \$1,000. It costs \$1 to buy a ticket. Find the expected value.

Earlier, we calculated the probability of matching all 6 numbers and the probability of matching 5 numbers:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815 \text{ for all 6 numbers,}$$

$$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205 \text{ for 5 numbers.}$$

Our probabilities and outcome values are:

Outcome	Probability of outcome
\$999,999	$\frac{1}{12271512}$
\$999	$\frac{252}{12271512}$
-\$1	$1 - \frac{253}{12271512} = \frac{12271259}{12271512}$

The expected value, then is:

$$(\$999,999) \times \frac{1}{12271512} + (\$999) \times \frac{252}{12271512} + (-\$1) \times \frac{12271259}{12271512} \approx -\$0.898$$

On average, one can expect to lose about 90 cents on a lottery ticket. Of course, most players will lose \$1.

In general, if the expected value of a game is negative, it is not a good idea to play the game, since on average you will lose money. It would be better to play a game with a positive expected value (good luck trying to find one!), although keep in mind that even if the *average* winnings are positive it could be the case that most people lose money and one very fortunate individual wins a great deal of money. If the expected value of a game is 0, we call it a **fair game**, since neither side has an advantage.

Not surprisingly, the expected value for casino games is negative for the player, which is positive for the casino. It must be positive or they would go out of business. Players just need to keep in mind that when they play a game repeatedly, their expected value is negative. That is fine so long as you enjoy playing the game and think it is worth the cost. But it would be wrong to expect to come out ahead.

Try it Now

A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him \$1. If any two dice match values, you get \$2. What is the expected value of this game? Would you play?

Expected value also has applications outside of gambling. Expected value is very common in making insurance decisions.

Example 11

According to the estimator at numericalexample.com, a 40-year-old man in the US has a 0.242% risk of dying during the next year. An insurance company charges \$275 for a life-insurance policy that pays a \$100,000 death benefit. What is the expected value for the person buying the insurance?

The probabilities and outcomes are

Outcome	Probability of outcome
\$100,000 - \$275 = \$99,725	0.00242
-\$275	1 - 0.00242 = 0.99758

The expected value is $(\$99,725)(0.00242) + (-\$275)(0.99758) = -\$33$.

Not surprisingly, the expected value is negative; the insurance company can only afford to offer policies if they, on average, make money on each policy. They can afford to pay out the occasional benefit because they offer enough policies that those benefit payouts are balanced by the rest of the insured people.

For people buying the insurance, there is a negative expected value, but there is a security that comes from insurance that is worth that cost.

10. Probability Reading I

Introduction

The probability of a specified event is the chance or likelihood that it will occur. There are several ways of viewing probability. One would be **experimental** in nature, where we repeatedly conduct an experiment. Suppose we flipped a coin over and over and over again and it came up heads about half of the time; we would expect that in the future whenever we flipped the coin it would turn up heads about half of the time. When a weather reporter says “there is a 10% chance of rain tomorrow,” she is basing that on prior evidence; that out of all days with similar weather patterns, it has rained on 1 out of 10 of those days.

Another view would be **subjective** in nature, in other words an educated guess. If someone asked you the probability that the Seattle Mariners would win their next baseball game, it would be impossible to conduct an experiment where the same two teams played each other repeatedly, each time with the same starting lineup and starting pitchers, each starting at the same time of day on the same field under the precisely the same conditions. Since there are so many variables to take into account, someone familiar with baseball and with the two teams involved might make an educated guess that there is a 75% chance they will win the game; that is, if the same two teams were to play each other repeatedly under identical conditions, the Mariners would win about three out of every four games. But this is just a guess, with no way to verify its accuracy, and depending upon how educated the educated guesser is, a subjective probability may not be worth very much.

We will return to the experimental and subjective probabilities from time to time, but in this course we will mostly be concerned with **theoretical** probability, which is defined as follows: Suppose

there is a situation with n **equally likely** possible outcomes and that m of those n outcomes correspond to a particular event; then the **probability** of that event is defined as $\frac{m}{n}$.

Basic Concepts

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology.

Events and Outcomes

The result of an experiment is called an **outcome**.

An **event** is any particular outcome or group of outcomes.

A **simple event** is an event that cannot be broken down further

The **sample space** is the set of all possible simple events.

Example 1

If we roll a standard 6-sided die, describe the sample space and some simple events.

Solution

The sample space is the set of all possible simple events:
 $\{1,2,3,4,5,6\}$



Some examples of simple events:

- We roll a 1
- We roll a 5

Some compound events:

- We roll a number bigger than 4
- We roll an even number

Basic Probability

Given that all outcomes are equally likely, we can compute the probability of an event E using this formula:

$$P(E) = \frac{\text{Number of outcomes corresponding to the event } E}{\text{Total number of equally likely outcomes}}$$

Example 2

If we roll a 6-sided die, calculate

1. $P(\text{rolling a 1})$
2. $P(\text{rolling a number bigger than 4})$

Solution

Recall that the sample space is $\{1,2,3,4,5,6\}$

1. There is one outcome corresponding to “rolling a 1,”
so the probability is $\frac{1}{6}$
2. There are two outcomes bigger than a 4, so the

$$\text{probability is } \frac{2}{6} = \frac{1}{3}$$

Probabilities are essentially fractions, and can be reduced to lower terms like fractions.

Example 3

Let's say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

Solution

There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is $\frac{14}{20} = \frac{7}{10}$.

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we

assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

Try it Now

At some random moment, you look at your clock and note the minutes reading.

1. What is probability the minutes reading is 15?
2. What is the probability the minutes reading is 15 or less?

Cards

A standard deck of 52 playing cards consists of four **suits** (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different **rank**: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

Example 4

Compute the probability of randomly drawing one card from a deck and getting an Ace.

Solution

There are 52 cards in the deck and 4 Aces so

$$P(\text{Ace}) = \frac{4}{52} = \frac{1}{13} \approx 0.0769$$

We can also think of probabilities as percents: There is a 7.69% chance that a randomly selected card will be an Ace.

Notice that the smallest possible probability is 0—if there are no outcomes that correspond with the event. The largest possible probability is 1—if all possible outcomes correspond with the event.

Certain and Impossible Events

An impossible event has a probability of 0.

A certain event has a probability of 1.

The probability of any event must be

$$0 \leq P(E) \leq 1$$

In the course of this chapter, if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.

Working with Events

Complementary Events

Now let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is $P(\text{six}) = \frac{1}{6}$. Now consider the probability that we do not roll a six: there are 5 outcomes that are not a six, so the answer is $P(\text{not a six}) = \frac{5}{6}$. Notice that

$$P(\text{six}) + P(\text{not a six}) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$$

This is not a coincidence. Consider a generic situation with n possible outcomes and an event E that corresponds to m of these outcomes. Then the remaining $n - m$ outcomes correspond to E not happening, thus

$$P(\text{not } E) = \frac{n - m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - \frac{m}{n} = 1 - P(E)$$

Complement of an Event

The **complement** of an event is the event “E doesn’t happen”

The notation \bar{E} is used for the complement of event E.

We can compute the probability of the complement using $P(\bar{E}) = 1 - P(E)$

Notice also that $P(E) = 1 - P(\bar{E})$

Example 5

If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

Solution

There are 13 hearts in the deck, so

$$P(\text{heart}) = \frac{13}{52} = \frac{1}{4}.$$

The probability of not drawing a heart is the complement:

$$P(\text{not heart}) = 1 - P(\text{heart}) = 1 - \frac{1}{4} = \frac{3}{4}$$

Probability of Two Independent Events

Example 6

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin and a 6 on the die.

Solution

We could list all possible outcomes:
 $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$.

Notice there are $2 \cdot 6 = 12$ total outcomes. Out of these, only 1 is the desired outcome, so the probability is $\frac{1}{12}$.

The prior example was looking at two independent events.

Independent Events

Events A and B are **independent events** if the probability of Event B occurring is the same whether or not Event A occurs.

Example 7

Are these events independent?

1. A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.
2. The two events (1) “It will rain tomorrow in Houston” and (2) “It will rain tomorrow in Galveston” (a city near Houston).
3. You draw a card from a deck, then draw a second card without replacing the first.

Solution

1. The probability that a head comes up on the second toss is $1/2$ regardless of whether or not a head came up on the first toss, so these events are independent.
2. These events are not independent because it is

more likely that it will rain in Galveston on days it rains in Houston than on days it does not.

3. The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

$P(A \text{ and } B)$ for Independent Events

If events A and B are independent, then the probability of both A and B occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

where $P(A \text{ and } B)$ is the probability of events A and B both occurring, $P(A)$ is the probability of event A occurring, and $P(B)$ is the probability of event B occurring

If you look back at the coin and die example from earlier, you can see how the number of outcomes of the first event multiplied by the number of outcomes in the second event multiplied to equal the total number of possible outcomes in the combined event.

Example 8

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability both are white?

Solution

The probability of choosing a white pair of socks is $\frac{6}{10}$.

The probability of choosing a white tee shirt is $\frac{3}{7}$.

The probability of both being white is

$$\frac{6}{10} \cdot \frac{3}{7} = \frac{18}{70} = \frac{9}{35}.$$

Try it Now

A card is pulled a deck of cards and noted. The card is then replaced, the deck is shuffled, and a second card is

removed and noted. What is the probability that both cards are Aces?

The previous examples looked at the probability of *both* events occurring. Now we will look at the probability of *either* event occurring.

Example 9

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin *or* a 6 on the die.

Solution

Here, there are still 12 possible outcomes:
{H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6}

By simply counting, we can see that 7 of the outcomes have a head on the coin *or* a 6 on the die *or* both—we use *or* inclusively here (these 7 outcomes are H1, H2, H3, H4, H5, H6, T6), so the probability is $\frac{7}{12}$. How could we have found this from the individual probabilities?

As we would expect, $\frac{1}{2}$ of these outcomes have a head,

and $\frac{1}{6}$ of these outcomes have a 6 on the die. If we add these, $\frac{1}{2} + \frac{1}{6} = \frac{6}{12} + \frac{2}{12} = \frac{8}{12}$, which is not the correct probability. Looking at the outcomes we can see why: the outcome H6 would have been counted twice, since it contains both a head and a 6; the probability of both a head *and* rolling a 6 is $\frac{1}{12}$.

If we subtract out this double count, we have the correct probability: $\frac{8}{12} - \frac{1}{12} = \frac{7}{12}$.

$$P(A \text{ or } B)$$

The probability of either A or B occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example 10

Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

$$P(\text{King or Queen}) = \frac{8}{52}$$

Note that in this case, there are no cards that are both a Queen and a King, so $P(\text{King and Queen}) = 0$.

Using our probability rule, we could have said:

$$P(\text{King or Queen}) = P(\text{King}) + P(\text{Queen}) - P(\text{King and Queen}) = \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52}$$

In the last example, the events were **mutually exclusive**, so $P(A \text{ or } B) = P(A) + P(B)$.

Example 11

Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

Half the cards are red, so
$$P(\text{Red}) = \frac{26}{52}$$

There are four kings, so $P(\text{King}) = \frac{4}{52}$

There are two red kings, so
 $P(\text{Red and King}) = \frac{2}{52}$

We can then calculate

$$P(\text{Red or King}) = P(\text{Red}) + P(\text{King}) - P(\text{Red and King}) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52}$$

Try it Now

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you reach in and randomly grab a pair of socks and a tee shirt, what the probability at least one is white?

Example 12

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

1. Has a red car *and* got a speeding ticket
2. Has a red car *or* got a speeding ticket.

Solution

	Speeding Ticket	No Speeding Ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is

$$\frac{15}{665} \approx 0.0226.$$

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people. So the probability is $\frac{195}{665} \approx 0.2932$.

We also could have found this probability by: $P(\text{had a red car}) + P(\text{got a speeding ticket}) - P(\text{had a red car and got a speeding ticket}) = \frac{150}{665} + \frac{60}{665} - \frac{15}{665} = \frac{195}{665}$.

Conditional Probability

Often it is required to compute the probability of an event given that another event has occurred.

Example 13

What is the probability that two cards drawn at random from a deck of playing cards will both be aces?

It might seem that you could use the formula for the probability of two independent events and simply multiply $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$. This would be incorrect, however, because the two events are not independent. If the first card drawn is an ace, then the probability that the second card is also an ace would be lower because there would only be three aces left in the deck.

Once the first card chosen is an ace, the probability that the second card chosen is also an ace is called the **conditional probability** of drawing an ace. In this case the “condition” is that the first card is an ace. Symbolically, we write this as: $P(\text{ace on second draw} \mid \text{an ace on the first draw})$.

The vertical bar “ \mid ” is read as “given,” so the above expression is short for “The probability that an ace is drawn on the second draw given that an ace was drawn on the first draw.” What is this probability? After an ace is drawn on the first draw, there are 3 aces out of 51 total cards left. This means that the conditional probability of drawing an

ace after one ace has already been drawn is $\frac{3}{51} = \frac{1}{17}$.

Thus, the probability of both cards being aces is

$$\frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}.$$

Conditional Probability

The probability the event B occurs, given that event A has happened, is represented as

$$P(B | A)$$

This is read as “the probability of B given A ”

Example 14

Find the probability that a die rolled shows a 6, given that a flipped coin shows a head.

These are two independent events, so the probability of

the die rolling a 6 is $\frac{1}{6}$, regardless of the result of the coin flip.

Example 15

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

1. Has a speeding ticket *given* they have a red car
2. Has a red car *given* they have a speeding ticket

Solution

	Speeding Ticket	No Speeding Ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

1. Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so

$$P(\text{ticket} \mid \text{red car}) = \frac{15}{150} = \frac{1}{10} = 0.1$$

2. Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so

$$P(\text{red car} \mid \text{ticket}) = \frac{15}{60} = \frac{1}{4} = 0.25$$

Notice from the last example that $P(B \mid A)$ is **not** equal to $P(A \mid B)$.

These kinds of conditional probabilities are what insurance companies use to determine your insurance rates. They look at the conditional probability of you having accident, given your age, your car, your car color, your driving history, etc., and price your policy based on that likelihood.

Conditional Probability Formula

If Events A and B are not independent, then

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$$

Example 16

If you pull 2 cards out of a deck, what is the probability that both are spades?

Solution

The probability that the first card is a spade is $\frac{13}{52}$.

The probability that the second card is a spade, given the first was a spade, is $\frac{12}{51}$, since there is one less spade in the deck, and one less total cards.

The probability that both cards are spades is

$$\frac{13}{52} \cdot \frac{12}{51} = \frac{156}{2652} \approx 0.0588$$

Example 17

If you draw two cards from a deck, what is the probability that you will get the Ace of Diamonds and a black card?

Solution

You can satisfy this condition by having Case A or Case B, as follows:

- Case A: you can get the Ace of Diamonds first and then a black card or
- Case B: you can get a black card first and then the Ace of Diamonds.

Let's calculate the probability of Case A. The probability that the first card is the Ace of Diamonds is $\frac{1}{52}$. The probability that the second card is black given that the first card is the Ace of Diamonds is $\frac{26}{51}$ because 26 of the remaining 51 cards are black. The probability is therefore $\frac{1}{52} \cdot \frac{26}{51} = \frac{1}{102}$.

Now for Case B: the probability that the first card is black is $\frac{26}{52} = \frac{1}{2}$. The probability that the second card is the

Ace of Diamonds given that the first card is black is $\frac{1}{51}$.

The probability of Case B is therefore $\frac{1}{2} \cdot \frac{1}{51} = \frac{1}{102}$, the same as the probability of Case 1.

Recall that the probability of A or B is $P(A) + P(B) - P(A \text{ and } B)$. In this problem, $P(A \text{ and } B) = 0$ since the first card

cannot be the Ace of Diamonds and be a black card.

Therefore, the probability of Case A or Case B is

$\frac{1}{101} + \frac{1}{101} = \frac{2}{101}$. The probability that you will get the Ace of Diamonds and a black card when drawing two cards from a deck is $\frac{2}{101}$.

Try it Now

In your drawer you have 10 pairs of socks, 6 of which are white. If you reach in and randomly grab two pairs of socks, what is the probability that both are white?

Example 18

A home pregnancy test was given to women, then pregnancy was verified through blood tests. The following table shows the home pregnancy test results. Find

1. $P(\text{not pregnant} \mid \text{positive test result})$
2. $P(\text{positive test result} \mid \text{not pregnant})$

Solution

	Positive Test	Negative Test	Total
Pregnant	70	4	74
Not Pregnant	5	14	19
Total	75	18	93

1. Since we know the test result was positive, we're limited to the 75 women in the first column, of which 5 were not pregnant. $P(\text{not pregnant} \mid \text{positive test result}) = \frac{5}{75} \approx 0.067$.
2. Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test. $P(\text{positive test result} \mid \text{not pregnant}) = \frac{5}{19} \approx 0.263$.

The second result is what is usually called a false positive: A positive result when the woman is not actually pregnant.

PART VI

MODULE 5: MODULAR ARITHMETIC

II. Modular Arithmetic Readings

For this first reading, you will need to access several external webpages provided by Khan Academy. Once you have completed all of the readings and practices, click the **Next** button below to continue the reading for Unit 1.

- What is Modular Arithmetic?
- Practice—Using the Modulo Operator
- Congruence Modulo
- Congruence Relation Practice
- Equivalence Relations
- The Quotient Remainder Theorem
- Modular Addition and Subtraction
- Practice—Modular Addition

12. Modular Arithmetic

Introduction to modular arithmetic using clocks and telling the time.



A YouTube element has been excluded from this version of the text. You can view it online here:

[https://library.achievingthedream.org/
hostosintrocollegemath/?p=34](https://library.achievingthedream.org/hostosintrocollegemath/?p=34)

13. Calculator Shortcut for Modular Arithmetic

Modular arithmetic

If you think back to doing division with whole numbers, you may remember finding the whole number result and the remainder after division.

Modulus

The **modulus**¹ is another name for the remainder after division.

For example, $17 \bmod 5 = 2$, since if we divide 17 by 5, we get 3 with remainder 2.

Modular arithmetic is sometimes called clock arithmetic, since analog clocks wrap around times past 12, meaning they work on

1. Sometimes, instead of seeing $17 \bmod 5 = 2$, you'll see $17 \equiv 2 \pmod{5}$. The \equiv symbol means “congruent to” and means that 17 and 2 are equivalent, after you consider the modulus 5.

a modulus of 12. If the hour hand of a clock currently points to 8, then in 5 hours it will point to 1. While $8 + 5 = 13$, the clock wraps around after 12, so all times can be thought of as modulus 12. Mathematically, $13 \bmod 12 = 1$.

Example 1

Compute the following:

1. $10 \bmod 3$
2. $15 \bmod 5$
3. $27 \bmod 5$

Answers

1. Since 10 divided by 3 is 3 with remainder 1, $10 \bmod 3 = 1$
2. Since 15 divided by 5 is 3 with no remainder, $15 \bmod 5 = 0$
3. $2^7 = 128$. 128 divide by 5 is 25 with remainder 3, so $2^7 \bmod 5 = 3$

Try it Now

Compute the following:

1. $23 \bmod 7$
2. $15 \bmod 7$
3. $2034 \bmod 7$

Recall that when we divide 17 by 5, we could represent the result as 3 remainder 2, as the mixed number $3\frac{2}{5}$,

or as the decimal 3.4. Notice that the modulus, 2, is the same as the numerator of the fractional part of the mixed number, and that the decimal part 0.4 is equivalent to the fraction $\frac{2}{5}$.

We can use these conversions to calculate the modulus of not-too-huge numbers on a standard calculator.

Modulus on a Standard Calculator

To calculate $a \bmod n$ on a standard calculator

1. Divide a by n
2. Subtract the whole part of the resulting quantity
3. Multiply by n to obtain the modulus

Example 2

Calculate $31345 \bmod 419$.

Answer

$$31345 \div 419 = 74.8090692$$

Now subtract
74 to get just
the decimal
remainder

$$74.8090692 - 74 = 0.8090692$$

Multiply this
by 419 to get
the modulus

$$0.8090692 \times 419 = 339$$

This tells us
0.8090692 was
equivalent to
339

419

In the text above, only a portion of the decimal value was written down. In practice, you should try to avoid writing down the intermediary steps, and instead allow your calculator to retain as many decimal values as it can.

PART VII

MODULE 6: MEASUREMENT

14. Measurement

<https://nrocnetwork.org/resources/courses/nroc-math/nroc-math-standard/>

Module 6 contains Measurement assessments